

Manuscript Number: EJOR-D-10-02318R2

Title: Sequential Market Entries and Competition Modelling in Multi-Innovation Diffusions

Article Type: Theory and Methodology Paper

Section/Category: Marketing

Keywords: marketing;
strategic planning;
synchronic and diachronic competition

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Manuscript Region of Origin: ITALY

Abstract: The diffusion of innovations for simultaneous processes cannot take into account and properly explain systematic perturbations due to competition-substitution effects if they are examined one by one. A first aspect in simultaneous competing diffusions is the distinction between simultaneous market entries (synchronic competition) and sequential entries (diachronic competition). In the latter case, the beginning of competition may upset the first entrant's diffusion. A second important aspect in multiple competition is represented by the choice to model the word-of-mouth effect either at the category level (balanced model) or at the brand level, separating the within-brand effect from the cross-brand one (unbalanced model). In this paper, balanced models are studied, and we propose a model that allows for a change in the parameter values of the first entrant as soon as the second one enters the market. The resulting differential system has a closed-form solution that enables, through sales data, an empirical validation of the assumptions underlying the model structure, improving the forecasting accuracy. An application to pharmaceutical drug competition is discussed.

List of changes on ‘Sequential Market Entries and Competition Modelling in Multi-Innovation Diffusions’

(paper EJOR-D-10-02318R2 w.r.t. EJOR-D-10-02318R1)

- Some sentences in the introductory section were simplified (as suggested by Referee #2).
- Section 4 (with the application) was split in two subsections. The first one is the older Section 4. Conversely, the short subsection 4.2 contains the comparison required by Referee #4.
- The Appendix on Cellular Automata has been included again (as suggested by Referee #4) as Appendix C. It is briefly mentioned in the outline of the paper at the end of Section 1 and it is pointed at the end of Section 5. The content of the Appendix is similar to the one included in the first version of our paper (EJOR-D-10-02318, December 2010). However, some sentences were added at the beginning to motivate its connection with the rest of the paper. The presentation has also been simplified in order to remove some redundant technical details.
- Two references had to be added.

**Highlights for ‘Sequential Market Entries and Competition Modelling in Multi-Innovation Diffusions’
(paper EJOR-D-10-02318R1)**

- ▷ We analyse a new diffusion of innovations model for sequential market entries.
- ▷ The beginning of competition may upset the first entrant’s diffusion parameters.
- ▷ The word-of-mouth effect spreads within a homogeneous category.
- ▷ The resulting differential system has a closed-form solution.
- ▷ An application to pharmaceutical competition is discussed comparing alternative models.

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7 Sequential Market Entries and Competition Modelling
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17 **Abstract**
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20 The diffusion of innovations for simultaneous processes cannot take into ac-
21 count and properly explain systematic perturbations due to competition-
22 substitution effects if they are examined one by one. A first aspect in simul-
23 taneous competing diffusions is the distinction between simultaneous market
24 entries (*synchronic* competition) and sequential entries (*diachronic* compe-
25 tition). In the latter case, the beginning of competition may upset the first
26 entrant's diffusion. A second important aspect in multiple competition is
27 represented by the choice to model the word-of-mouth effect either at the
28 category level (*balanced* model) or at the brand level, separating the within-
29 brand effect from the cross-brand one (*unbalanced* model). In this paper, bal-
30 anced models are studied, and we propose a model that allows for a change
31 in the parameter values of the first entrant as soon as the second one en-
32 ters the market. The resulting differential system has a closed-form solution
33 that enables, through sales data, an empirical validation of the assumptions
34 underlying the model structure, improving the forecasting accuracy. An ap-
35 plication to pharmaceutical drug competition is discussed.
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46 *Keywords:* marketing, strategic planning, synchronic and diachronic
47 competition
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51 **1. Introduction**
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54 The diffusion of competing products within the same marketplace is an
55 important question for marketing experts. The different products (brands)
56 that represent a category are often similar enough to compete for the same
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7 group of potential adopters. A few competitors may give rise to the category
8 from the beginning. Conversely, a more common situation is observed when
9 the category stems from a monopoly of a pioneering brand followed by com-
10 petitors. Sequential market entries, or *diachronic* competition, originate the
11 ‘regime change’ problem for the previously entered pioneers. Models that
12 do not allow for a change in the market’s structure due to the beginning of
13 competition perform poorly, in some cases, because the global fit is only a
14 ‘compromise’ between two very different processes (the before-competition
15 evolution and the after-competition one).
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22 A specific aspect of multi-product growth modelling is related to inter-
23 personal communication effects due to word-of-mouth (w.o.m.). The inter-
24 personal influence of adopters may be brand-specific (potential users join a
25 brand only as a result of communication with customers of that brand) or it
26 may be a function of the product class knowledge (word-of-mouth is spread
27 by customers of the whole category). For the latter case, a model where
28 the word-of-mouth effect does not separate the adoptions of each brand from
29 those of the competitors (*balanced* model) is suitable and correctly motivated
30 by the homogeneity of the product category. However, in heterogeneous situ-
31 ations, the relative knowledge may be decomposed into brand-specific factors
32 (Peterson and Mahajan, 1978; Kalish et al., 1995; Mahajan et al., 1993). *Un-*
33 *balanced* models distinguish within-brand from cross-brand w.o.m. effects.
34 For example, the paper by Savin and Terwiesch (2005) proposes the decom-
35 position of the category’s relative knowledge in brand-specific (unbalanced)
36 components. Libai et al. (2009a) also address the topic with a model that is
37 a special case of the Savin and Terwiesch (2005) model. Unbalanced models
38 for heterogeneous competing brands are not the aim of this paper and will
39 be studied in the future.
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52 Within balanced models, Krishnan et al. (2000) introduce a model for the
53 diachronic case (hereafter called KBKD, Krishnan Bass Kumar Diachronic
54 model), which considers the late introduction of a third competitor with
55 respect to the two previously existing synchronic actors in the same category.
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7 This model allows for a *partial* regime change in the parameter values and
8 assumes unnecessary constraints.
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10 In this paper, a new model, Competition and Regime Change Diachronic
11 (CRCDD), is introduced, focusing on the reasons for a regime change of the
12 previously existing products. This model allows for a change in *all* the param-
13 eter values after competition. The CRCDD model will be examined, to obtain
14 a closed-form solution including KBKD results as special cases. Moreover,
15 in Appendix A, we study the competition under common exogenous inter-
16 ventions (regulations, incentives, marketing strategies, economic cycles, etc.)
17 that affect the joint evolution.
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20 The paper is organised as follows. Section 2 provides the related back-
21 ground. In Section 3, we study the basic differential characterisations of
22 the proposed twofold *diachronic* case, CRCDD, and compare its closed-form
23 solution with the nested KBKD solution by Krishnan et al. (2000). In Sec-
24 tion 4, we consider a specific application of the CRCDD model with reference
25 to competing pharmaceutical drugs. Final remarks and the discussion are
26 presented in Section 5. In Appendix A, we consider the problem of competi-
27 tion under environmental or exogenous intervention, for both synchronic and
28 diachronic cases, in order to extend our results following the inspiration of
29 Bass et al. (1994). Finally, in Appendix C, we illustrate the link between the
30 proposed model and a Cellular Automata system, which may be appropriate
31 to describe how an agent performs the choice among different alternatives
32 (see the underlying reasons summarised in Section 5).
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46 **2. Background**

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48 The diffusion of innovations in a social system has been examined in the past
49 four decades from many points of view. Sociologists (in particular Rogers,
50 2003), mathematicians, physicists, quantitative marketing experts, statisti-
51 cians, biologists, and epidemiologists have made substantial contributions to
52 the theory and have realised widespread applications in different fields. For
53 recent reviews, within quantitative marketing, see Meade and Islam (2006)
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7 and Peres et al. (2010).

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9 The main effort expressed in the literature is usually concentrated on uni-
10 variate versions of diffusion processes. The pioneering work by Bass (1969),
11 and its subsequent extensions are valuable modelling tools for forecasting
12 separate diffusion of innovations processes. The Bass Model (BM) relates
13 to homogeneous category-level sales growth or, more extensively, to specific
14 brand-level sales whose diffusion does not depend upon competing brands.
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18 We emphasise that the original BM was initially confined to a restric-
19 tive interpretation of diffusions referring to new durable products in order to
20 preserve a perfect correspondence between an agent and his/her single pur-
21 chase. Nevertheless, a broader interpretation, based on the same equation,
22 is equally possible if we refer to a different unit, namely, the simple adoption
23 as a relevant act in describing the sale. Agents may adopt more than once,
24 and each purchase may be classified either as an innovative or an imitative
25 action (avoiding classification of agents). In this sense, the first purchase and
26 repeated purchases may be described through the same model structure.
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33 An outstanding advance in introducing external control variables in the
34 diffusion of innovation dynamics is the definition of the Generalised Bass
35 Model (GBM, Bass et al., 1994). This extension allows the introduction of
36 exogenous covariates, in particular, relative prices and relative advertising
37 effects as a first relevant example for marketing, management sciences, and
38 operations research methodologies.
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43 Multi-product growth models were first examined in the marketing litera-
44 ture by Peterson and Mahajan (1978) under the assumption of simultaneous
45 (*synchronic*) launches. They classify co-existing products in the marketplace
46 into four categories: independent, complementary, contingent, and substitute
47 products. Only substitute products generate competition, which is modelled
48 through the introduction of within-brand and cross-brand word-of-mouth
49 (w.o.m.) effects related to *brand-specific* residual markets. This choice par-
50 tially contradicts the definition of a *common* category product where substi-
51 tution and related competition are generated.
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7 Category level diffusions originate mainly from a monopolistic diffusion
8 process. New entrants generate competition and substitution effects, which
9 may be described, following Parker and Gatignon (1994), in two classes. The
10 first one is composed of diffusion processes, which are product class driven,
11 i.e., there is a common residual market for both products, which is obtained
12 as the difference between the initial market potential and the past *category*
13 sales. The second class, conversely, is formed by diffusion processes that are
14 specific to the individual brand (i.e., each competitor retains its own market)
15 as in Peterson and Mahajan (1978).
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22 Further contributions to the multi-product growth models may be found
23 in Mahajan et al. (1993). The restricted version of this model is based on
24 diffusion processes that are product class driven. This assumption correctly
25 represents the situation of multiple substitute products in the same market-
26 place. However, in this model interpersonal influence is brand-specific, which
27 partially contradicts previous product class concepts. Moreover, the innova-
28 tive effect is proportional to the specific residual market. Similar models are
29 presented in Kalish et al. (1995), Yan and Ma (2011). A different perspec-
30 tive in analysing competition is examined, for instance, in Krishnamoorthy
31 et al. (2010) in the synchronic duopolistic case. The proposed dynamics
32 mimic a GBM with two strong restrictions in the evolutionary parameters.
33 For both brands, $p = q = 1$, and therefore the resulting dynamics are ex-
34 clusively driven by prices and advertising factors. This framework allows
35 optimal policies but excludes well-known effects of innovators and imitators
36 based on more complex interpersonal utility criteria.
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47 A different approach in model building rests on a brand-specific dynamic
48 residual market that generalises (even if in a synchronic context) the Lotka-
49 Volterra framework (Abramson and Zanette, 1998; Morris and Pratt, 2003;
50 Tang and Zhang, 2005).
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53 As mentioned in the introductory section, we focus on a homogeneous
54 product category where a *common* residual market is appropriately defined
55 for perfect substitute products. Within this context, an attempt to model
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7 simultaneous (synchronic) competition between two brands within the same
8 market was performed in Bonaldo (1991). The proposed balanced model
9 splits the rate sales into two separate equations under the hypothesis that the
10 category is sufficiently homogeneous and, therefore, the relative knowledge of
11 the product class is common knowledge driving word-of-mouth. Obviously,
12 parameters that modulate access to that knowledge may be different for the
13 two products. Krishnan et al. (2000) propose the same version of the Bonaldo
14 (1991) model for the synchronic case, and introduce an original representation
15 for the diachronic case that, in the next section, will be compared, in more
16 detail, with the new model introduced in this paper.
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25 **3. Twofold diachronic competition, the CRCD model**

26 *3.1. Definition*

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29 Infrequently observed are two or more diffusion processes that are exactly
30 synchronic and with direct effects of substitution between competitors within
31 the same environment. On the contrary, it is a common experience to observe
32 the late entrance of new diffusion processes. Let us consider a simpler twofold
33 case with the late entrance of the second competitor at time $t = c_2$ with
34 $c_2 > 0$, where $t = 0$ denotes the time origin for the first competitor. We
35 propose here a new model focusing on Competition and Regime Change of
36 the first entrant in Diachronic processes, namely CRCD:
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$$\begin{aligned}
 z_1'(t) &= m \left\{ \left[p_{1a} + q_{1a} \frac{z(t)}{m} \right] (1 - I_{t>c_2}) + \left[p_{1c} + q_{1c} \frac{z(t)}{m} \right] I_{t>c_2} \right\} \left[1 - \frac{z(t)}{m} \right] \\
 z_2'(t) &= m \left[p_2 + q_2 \frac{z(t)}{m} \right] \left[1 - \frac{z(t)}{m} \right] I_{t>c_2} \\
 m &= m_a(1 - I_{t>c_2}) + m_c I_{t>c_2} \\
 z(t) &= z_1(t) + z_2(t) I_{t>c_2},
 \end{aligned} \tag{1}$$

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53 where $z_i(t)$, $i = 1, 2$, denotes the cumulative sales of brand i at time t ,
54 $z_i'(t) = dz_i(t)/dt$ denotes the instantaneous (rate) sales, and $z(t) = z_1(t) +$
55 $z_2(t)$ represents the category cumulative diffusion process (aggregate sales).
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58 Moreover, in Equation (1), m denotes the limiting state of $z(t)$, as far as
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7 $t \rightarrow +\infty$, the constant aggregate carrying capacity or market potential.
8 Parameters p_i (q_i), $i = 1, 2$, refer to innovators (imitators), as in the standard
9 Bass Model, and I_E is the indicator function of the event E ($I_E = 1$, if E is
10 true, while $I_E = 0$ elsewhere).

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13 The residual market $m[1 - z(t)/m]$ is a subset of the whole category mar-
14 ket potential m , from which the sales of both brands have to be subtracted
15 *without penalties*. This model is adequate to describe competition between
16 perfect substitute products competing for the same group of adopters. Co-
17 herently, the mechanism which governs interpersonal influence is assumed
18 to be balanced and based on a *common driver*, that is, a common relative
19 knowledge $z(t)/m$, and, therefore, it is assumed to have a typical property
20 of a particular competitive *niche*: a competitive market based on a common
21 class of substitutes that ‘cooperate’ in defining the agents’ awareness towards
22 the product category. Notice that the fraction q_i of the common awareness
23 driver, $z(t)/m$, is a specific imitative characteristic of the i th diffusion pro-
24 cess.
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33 We underline that model (1) explicitly allows for a structural change at
34 the competition’s beginning. In the stand-alone period, $t \leq c_2$, $z(t)$ equals
35 $z_1(t)$, and hence, the first equation simplifies to
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$$38 \quad z'_1(t) = m_a \left[p_{1a} + q_{1a} \frac{z_1(t)}{m_a} \right] \left[1 - \frac{z_1(t)}{m_a} \right], \quad t \leq c_2. \quad (2)$$

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42 Conversely, after c_2 , $z(t)$ measures the sales of the whole category under
43 study, $z(t) = z_1(t) + z_2(t)$, and, therefore, we obtain from (1) that
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$$46 \quad z'_1(t) = m_c \left[p_{1c} + q_{1c} \frac{z_1(t) + z_2(t)}{m_c} \right] \left[1 - \frac{z_1(t) + z_2(t)}{m_c} \right], \quad t > c_2. \quad (3)$$

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49 Although in both cases $z(t)/m$ represents the relative category sales, the
50 comparison between (2) and (3) highlights that they describe *different model*
51 *structures*. The ‘usual’ constraints $p_{1a} = p_{1c}$ and $q_{1a} = q_{1c}$ that assume
52 *common parameter values* would appear as a contradictory choice. This is
53 the reason two different innovative parameters and two imitative parameters
54 are introduced for the first competitor in the CRCDC. For the ‘stand-alone’
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Table 1: Diachronic models. CRCD=current model, KBKD=Krishnan et al. (2000) model. PR1=first entrant product, PR2=second entrant product, AGG=aggregate model.

DIACHRONIC MODELS		BEFORE COMPETITION			UNDER COMPETITION		
		INN	IM	$t < c_2$ MKT POT	INN	$t \geq c_2$ IM	MKT POT
CRCD (# p. 8) (# p. 6)	PR1	p_{1a}	q_{1a}	m_a	p_{1c}	q_{1c}	m_c
	PR2	0	0		p_2	q_2	
	AGG	p_{1a}	q_{1a}		$p_{1c} + p_2$	$q_{1c} + q_2$	
KBKD (# p. 6) (# p. 5)	PR1	p_{1a}	q_{1a}	m_a	p_{1a}	q_{1c}	m_c
	PR2	0	0		0	q_2	
	AGG	p_{1a}	q_{1a}		p_{1a}	$q_{1c} + q_2$	

situation, we denote them by subscript a and, for the competitive situation, with the subscript c . We underline that, for this reason, p_{1a} and q_{1a} cannot be compared with p_{1c} and q_{1c} because they refer to different model structures. From a more general point of view, this feature allows dealing with the ‘regime change’ problem. The transition from a monopolistic market to a duopolistic one is likely to upset the first entrant’s diffusion structure and to give rise to different parameters for the first competitor and (or) to a new *category* carrying capacity, m_c . The comparison between m_a and m_c might be used only to describe the effect of the competition on the total size of the market niche.

As previously mentioned, the KBKD model by Krishnan et al. (2000) is a special case of the CRCD model obtained when two unnecessary constraints, $p_{1a} = p_{1c}$ and $p_2 = 0$, are introduced (see Table 1).

3.2. Solution

In Appendix B, it is proven that system (1) has the following solution:

$$z_1(t) = {}_a z_1(t)(1 - I_{t>c_2}) + {}_c z_1(t)I_{t>c_2} \quad (4)$$

$$z_2(t) = 0 \cdot (1 - I_{t>c_2}) + {}_c z_2(t)I_{t>c_2} = {}_c z_2(t)I_{t>c_2}, \quad (5)$$

where $p = p_{1c} + p_2$, $q = q_{1c} + q_2$, and

$${}_a z_1(t) = m_a \frac{1 - e^{-(p_{1a} + q_{1a})t}}{1 + \frac{q_{1a}}{p_{1a}} e^{-(p_{1a} + q_{1a})t}} \quad (6)$$

$${}_c z_1(t) = m_c \frac{q_{1c}}{q} w(t) + \frac{q_2}{q} z_s + m_c \frac{p}{q} \left(\frac{p_{1c}}{p} - \frac{q_{1c}}{q} \right) \ln y(t) \quad (7)$$

$${}_c z_2(t) = m_c \frac{q_2}{q} w(t) - \frac{q_2}{q} z_s + m_c \frac{p}{q} \left(\frac{p_2}{p} - \frac{q_2}{q} \right) \ln y(t) \quad (8)$$

$$w(t) = \frac{1 + \frac{q}{p} \frac{z_s}{m_c} - \left(1 - \frac{z_s}{m_c}\right) e^{-(p+q)(t-c_2)}}{1 + \frac{q}{p} \frac{z_s}{m_c} + \frac{q}{p} \left(1 - \frac{z_s}{m_c}\right) e^{-(p+q)(t-c_2)}} \quad (9)$$

$$y(t) = \frac{1 + \frac{q}{p}}{1 + \frac{q}{p} \frac{z_s}{m_c} + \frac{q}{p} \left(1 - \frac{z_s}{m_c}\right) e^{-(p+q)(t-c_2)}}. \quad (10)$$

The function $w(t)$ represents the relative aggregate sales under competition, ${}_c z(t)/m_c$. We remark that the aggregate sales in the whole range, resulting from the CRCDC model $z(t) = {}_a z(t)(1 - I_{t > c_2}) + {}_c z(t)I_{t > c_2}$, $t \in [0, +\infty)$, are not described by a pure Bass model. It is a two-regime function based on local Bass models: $BM(m_a, p_{1a}, q_{1a})$ for $t \in [0, c_2]$ and $BM(m_c, p, q)$ for $t \in (c_2, +\infty)$, with a continuity condition $z(c_2) = {}_a z(c_2) = z_s$ for $t = c_2$.

In (7) we can recognise three components: a ‘baseline’ process

$$b_1(t) = m_c \frac{q_{1c}}{q} w(t),$$

a time dependent ‘perturbation’

$$r_1(t) = m_c \frac{p}{q} \left(\frac{p_{1c}}{p} - \frac{q_{1c}}{q} \right) \ln y(t),$$

whose sign and size depend upon parameter values, and a constant term, $\frac{q_2}{q} z_s$. In a similar way, in (8) we can recognise three components: the ‘baseline’ process

$$b_2(t) = m_c \frac{q_2}{q} w(t),$$

the time dependent ‘perturbation’

$$r_2(t) = -r_1(t) = m_c \frac{p}{q} \left(\frac{p_2}{p} - \frac{q_2}{q} \right) \ln y(t),$$

and the constant term, $-\frac{q_2}{q}z_s$. The last two components exactly compensate for the corresponding ones seen in ${}_c z_1(t)$. For each product, the ‘baseline’ process represents a share of the whole category sales. The time dependent departures, $r_i(t)$, $i = 1, 2$, highlight competition/substitution effects and, in particular, these perturbations tell us whether competition generates an advantage ($q_{1c}/p_{1c} > q_2/p_2$) or a drawback ($q_{1c}/p_{1c} < q_2/p_2$) for the first competitor. Notice that, if local parameters are proportional, i.e., $q_{1c}/p_{1c} = q_2/p_2$, the competition effects, $r_i(t)$, $i = 1, 2$, vanish. Finally, the constant term $\frac{q_2}{q}z_s$ represents the advantage of the first entrant.

In Appendix A, we extend the balanced CRCDC model in order to take into account strategic, or marketing mix effects that may modify the evolutionary dynamics of both competitors through a common intervention function $x(t)$.

As a final remark, we underline that the CRCDC model can also be easily extended to deal with more than two competitors, and that closed-form solutions exist for the most complex cases where the entrance of each new competitor generates a regime change.

3.3. Properties

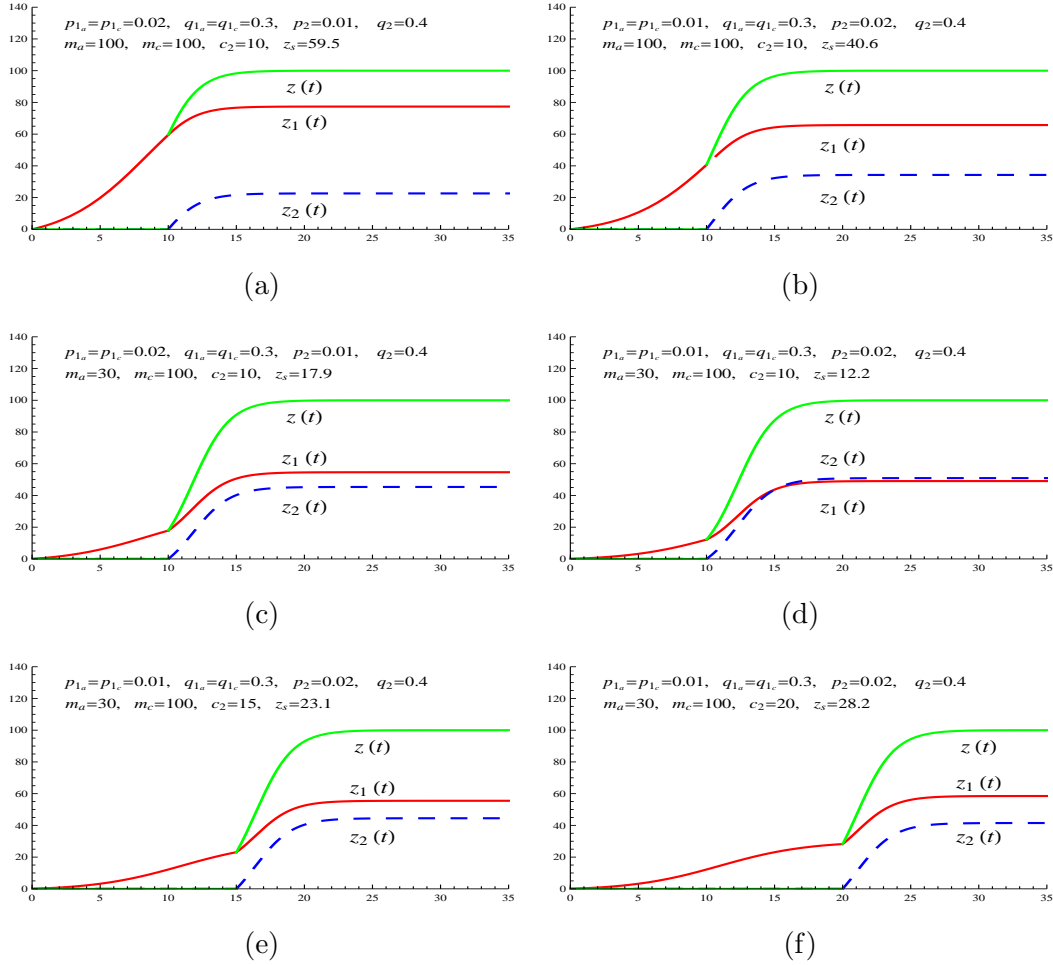
The asymptotic behaviour of equations (7) and (8) is straightforward and highlights a non-intuitive splitting of aggregate carrying capacity m_c , namely,

$$\begin{aligned} \lim_{t \rightarrow +\infty} {}_c z_1(t) &= m_c \frac{q_{1c}}{q} + m_c \frac{p}{q} \left(\frac{p_{1c}}{p} - \frac{q_{1c}}{q} \right) \ln \left(\frac{1 + q/p}{1 + (q/p)\frac{z_s}{m_c}} \right) + \frac{q_2}{q} z_s \\ \lim_{t \rightarrow +\infty} {}_c z_2(t) &= m_c \frac{q_2}{q} + m_c \frac{p}{q} \left(\frac{p_2}{p} - \frac{q_2}{q} \right) \ln \left(\frac{1 + q/p}{1 + (q/p)\frac{z_s}{m_c}} \right) - \frac{q_2}{q} z_s. \end{aligned} \quad (11)$$

The first equation, in particular, describes the potential sales for the first competitor. The comparison with m_a may be used to highlight the effect (either positive or negative) due to the late entrant.

The closed-form solution (6)–(8) also allows simple graphical analyses of competition structures consistent with the CRCDC model. Figure 1 summarises, with a qualitative description, some possible relationships. For the

Figure 1: Twofold diachronic competition, CRCD: comparison among different situations.



sake of simplicity, the situations here depicted describe the case of the first competitor's unmodified parameters after the second competitor's entrance. In Figure 1(a) we consider a new entrant at time $c_2 = 10$ with no incremental market potential, $m_a = m_c = 100$, with $p_{1a} = p_{1c} = 0.02$, $p_2 = 0.01$, $q_{1a} = q_{1c} = 0.3$, and $q_2 = 0.4$. We observe a diminished ceiling of the first competitor, $z_1(+\infty) \simeq 78$. If we interchange the values of parameters $p_{1a} = p_{1c}$ and p_2 , i.e., $p_{1a} = p_{1c} = 0.01$ and $p_2 = 0.02$, we note a worse situation for the first entrant (see Figure 1(b)). We may be interested in the effect of a market potential expansion. In Figure 1(c) we notice, for $m_a = 30$ and

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7 $m_c = 100$, an interesting benefit for the first entrant with an asymptotic
8 potential $z_1(+\infty) \simeq 54$, as compared with the stand-alone asymptotic level
9 (30). If we interchange the innovator parameters of the two competitors,
10 that is $p_{1a} = p_{1c} = 0.01$ and $p_2 = 0.02$, we observe a more limited benefit for
11 the first entrant (Figure 1(d)). In this situation the second entrant becomes
12 the market leader.
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17 Figure 1 may be helpful also to evaluate the effect of the time at which
18 the second brand enters the market, c_2 , on competition. The situation de-
19 scribed in Figure 1(d) depicts the entrance at time 10 of a very performant
20 brand that quickly overtakes the pioneering one. Figures 1(e) and 1(f) show
21 that (keeping all parameters constant with respect to case 1(d)) a delay in
22 competition's beginning, respectively at time $c_2 = 15$ and $c_2 = 20$, would re-
23 duce the advantage of the second entrant. The greater the interval between
24 launches, the larger is the advantage of the pioneering brand, and if the in-
25 terval between launches is too long, the gap cannot be closed even by a keen
26 competitor. Nevertheless, we can also observe that the difference between
27 case 1(d) and case 1(e) is greater than the difference between 1(e) and 1(f),
28 although they both correspond to a delay of 5 time units. The reason is
29 that the first entrant's advantage is conveyed by z_s (the cumulative sales of
30 the first brand when the second one is launched). The difference between
31 $z_s(c_2 = 10) = 12.2$ and $z_s(c_2 = 15) = 23.1$ is larger than the difference be-
32 tween $z_s(c_2 = 15) = 23.1$ and $z_s(c_2 = 20) = 28.2$. In other words, the delay
33 of the second entrant is always a backfire for it but, after the first brand
34 peaked, the damage decreases as time elapses. Similar conclusions were also
35 obtained with different values of the diffusion parameters of the two brands.
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48 We notice that the limits in (11) may also be studied as a function of
49 z_s . The optimal values of these functions would indicate that the maximum
50 long-term advantage/drawback of the two competitors depends on the time
51 launch of the second competitor. It should be noted, however, that the
52 optimal values depend upon parameter values which may not be known in
53 advance when the launch has to be planned.
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7 It is useful to examine more deeply the relationship between m_a and m_c .
8 In particular, in light of (11), a strong closeness between m_a and m_c , due
9 to the perfect knowledge of a mature environment, generates a substitutive
10 competition. Conversely, a large divergence between m_a and m_c is synony-
11 mous with a first exploratory situation, within which the competition effect
12 is a stimulating causal precursor with a possible benefit for the first entrant.
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15 In the special situation of a synchronic competition, where the competi-
16 tors enter into the market simultaneously (see Bonaldo, 1991 and Krishnan
17 et al., 2000) we obtain
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$$\begin{aligned}
 z_1(t) &= m \frac{q_1}{q} \frac{1 - e^{-(p+q)t}}{1 + \frac{q}{p} e^{-(p+q)t}} + m \frac{p}{q} \left(\frac{p_1}{p} - \frac{q_1}{q} \right) \ln \frac{1 + \frac{q}{p}}{1 + \frac{q}{p} e^{-(p+q)t}} \\
 z_2(t) &= m \frac{q_2}{q} \frac{1 - e^{-(p+q)t}}{1 + \frac{q}{p} e^{-(p+q)t}} + m \frac{p}{q} \left(\frac{p_2}{p} - \frac{q_2}{q} \right) \ln \frac{1 + \frac{q}{p}}{1 + \frac{q}{p} e^{-(p+q)t}},
 \end{aligned} \tag{12}$$

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29 where $p_1 = p_{1a} = p_{1c}$, $q_1 = q_{1a} = q_{1c}$, $m = m_a = m_c$.

30 We notice that the well-known Givon et al. (1995) paper about piracy
31 studies the legal and illegal trajectories of a common product. Only the legal
32 series is observable; the illegal one is latent. That model may be included as a
33 special case in (12) with the following constraints: $p_1 = p_\ell$, $p_2 = 0$, $q_1 = \alpha q_\ell$,
34 $q_2 = (1 - \alpha)q_\ell$. In this case, direct estimation of all involved parameters,
35 m , p_ℓ , q_ℓ , and α may be carried out through the legal observable series.
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43 **4. An application to pharmaceutical drugs**

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45 *Cimetidine* is a histamine H₂-receptor antagonist that inhibits the pro-
46 duction of acid in the stomach. It is largely used in the treatment of heart-
47 burn and peptic ulcers. Cimetidine was the culmination of a project at Smith,
48 Kline & French. It was approved and marketed in the UK in 1976 and in the
49 US starting January 1, 1979. Cimetidine became the first drug to reach more
50 than \$1 billion a year in sales, thus making it the first blockbuster drug.
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55 *Ranitidine* was developed by Glaxo in an effort to match the success
56 of Smith, Kline & French (prior to the merger of the two companies into
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Table 2: Multivariate estimation results for model CRCD.

	Estimate	Standard Error	95% Confidence Interval
m_a	9976.41	1370.42	(7254.63, 12698.19)
m_c	367163.35	41693.76	(284365.9, 449980.8)
p_{1a}	0.01218	0.00169	(0.00883, 0.01554)
q_{1a}	0.43866	0.06373	(0.31208, 0.56523)
p_{1c}	0.00180	0.00021	(0.00139, 0.00221)
q_{1c}	-0.00255	0.00021	(-0.00298, -0.00213)
p_2	0.00346	0.00025	(0.00295, 0.00396)
q_2	0.04091	0.00288	(0.03518, 0.04663)

$$R^2 = 0.999489$$

GlaxoSmithKline) with cimetidine. Ranitidine was found to have a far-improved tolerability profile (i.e., fewer adverse drug reactions), longer-lasting action, and 10 times the activity of cimetidine. Ranitidine was introduced in 1981 and was the world’s biggest-selling prescription drug by 1988.

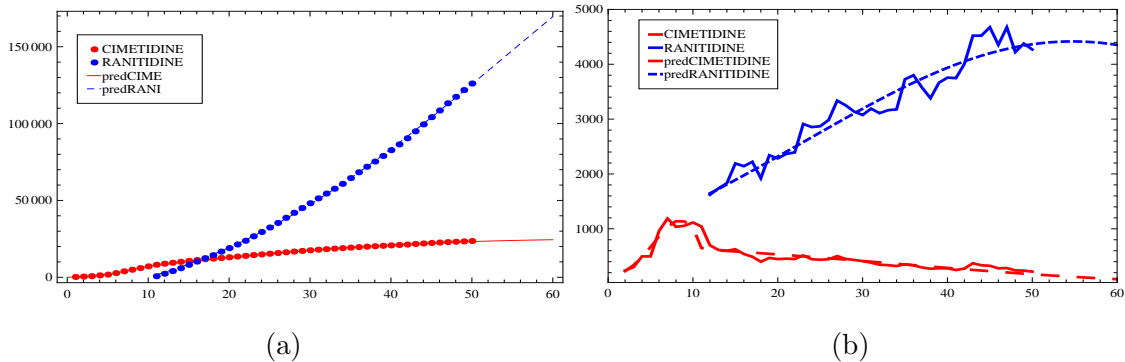
Here we examine the competition between cimetidine and ranitidine in the Italian pharmaceutical market. Cimetidine was introduced in the second quarter of 1979, while ranitidine was launched in the fourth quarter of 1981. Our data, provided by IMS-Health, Italy, consist of the cumulative quarterly number of packages of cimetidine and ranitidine sold in Italy. Data are available until the third quarter of 1991 (50 observations for cimetidine and 40 observations for ranitidine).

4.1. Comparison between CRCD and KBKD models

The equations stemming from system (1), i.e., (6), (7) (which together give rise to (4)), and (8), were fitted simultaneously, applying the Beauchamp & Cornell technique (B&C, Beauchamp and Cornell, 1966). In the first step the models (4) and (8) pertaining to the two products were fitted separately to their own series in order to estimate the covariance matrix for the two responses through residuals; at the second step, the models were fitted jointly with weighted nonlinear least squares using the covariance matrix as weight.

Parameter estimates are summarised in Table 2 and the agreement between

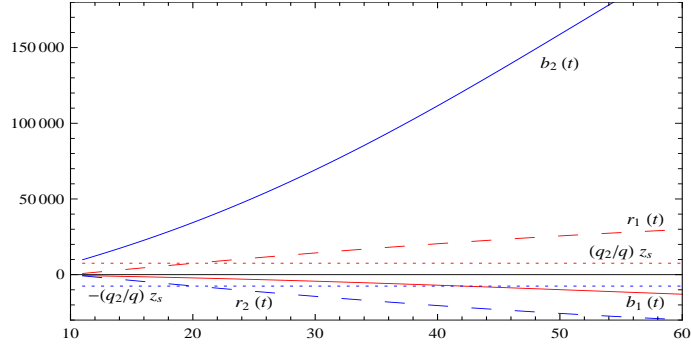
Figure 2: CRCD. Comparison between observed and fitted values: cumulative (a) and rate (b) sales.



observed and fitted values is shown in Figure 2. The most impressive feature is a huge increase in market potential after ranitidine’s launch. We underline that since we deal with consumables, i.e., with repeatedly purchased goods, parameters should be interpreted in a different way (see the comment about agents and their purchases in Section 2). In detail, m , the market potential, represents the total number of packages sold instead of the total number of customers. We argue, however, that the application of diffusion models to consumables is in any case correct, since an upper bound to consumption exists. We remark that m_a is estimated only through the stand-alone data of cimetidine’s series (the first 10 observations). For this reason it is well-known that the market potential might be underestimated. Moreover, we observe that differences between \hat{p}_{1a} and \hat{p}_{1c} and \hat{q}_{1a} and \hat{q}_{1c} are significant: both parameters are reduced after c_2 . In particular, large parameters p_{1a} and q_{1a} , together with a ‘small’ market potential, can be interpreted as an intensive diffusion cycle that exhausts itself in a short time period. On the other side, after competition, we observe a much bigger market potential which is approached by the actual sales in a slower and more regular way (small \hat{p}_{1c} and \hat{q}_{1c} values). The first effect of competition is thus a change in the structure of diffusion of the category under study. The first, imperfect active compound spread fast but was about to complete its history when the new one was launched. The new compound propagated in a larger competitive

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Figure 3: CRCD. Cumulative model components.



niche without slowdown signals.

We remark that the negative sign of \hat{q}_{1c} should not be interpreted as a non-identification symptom (the confidence interval is really narrow and the same value was also obtained with the direct estimation approach discussed below). Conversely, the sign of \hat{q}_{1c} should be read in the light of the whole model formulation. If we substitute the obtained estimates in model (1), we have, after ranitidine’s launch, that

$$z'_1(t) \propto 0.0018 - 0.00255 z(t)/m_c$$

$$z'_2(t) \propto 0.00346 + 0.04091 z(t)/m_c.$$

In other words, ranitidine exploited the word-of-mouth effect of the whole category and had a non-negligible innovative effect. Conversely, after the start of competition, cimetidine spread only thanks to the innovative component, experienced a negative w.o.m. (and was enhanced by the increase of the category market potential). We notice that the widespread opinion that ‘theoretically plausible restrictions’ (e.g., non-negative imitative effects) should be imposed is unconvincing because it would hide an essential observed feature of the competition.

The effect of competition between the two products can also be deduced by an examination of Figure 3, where the model components of the closed-form solutions after c_2 are plotted. Here the first competitor was completely

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7 crushed by ranitidine ($b_1(t) < 0$), but gained through a positive increasing
8 perturbation $r_1(t)$ and, obviously, through $(q_2/q)z_s$. In this case, cimetidine
9 was facing a declining phase of its diffusion process when ranitidine entered
10 the market. This upheaval dramatically increased the market of the whole
11 category and delayed the fade of cimetidine, granting it the ‘crumbs’ of the
12 category market. This fact is also confirmed by the comparison between \hat{m}_a
13 (9976) and the limit of ${}_c z_1(t)$ evaluated as in (11), that leads to 23918 (the
14 limit of ${}_c z_2(t)$ equals 343255).

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16 In order to compare the previous model with alternative solutions, the
17 same data were used to fit two independent Bass models (one for each se-
18 ries), and the joint model (KBKD) by Krishnan et al. (2000). The reason is
19 that the KBKD is the only alternative balanced diachronic model available
20 in the literature. Since the KBKD is nested within the CRCDC model, the
21 latter will obviously perform better. The aim is to see whether the improve-
22 ment is large enough to make the extension worthwhile. Unbalanced models
23 were excluded from the comparison for two reasons. Firstly, the drugs here
24 analysed represent a very homogeneous category for which an unbalanced
25 w.o.m. specification appears, to our belief, redundant. Secondly, we aimed
26 at showing the relevance of the regime change option within balanced models
27 that describe the w.o.m. effect in a similar way.

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29 Moreover, in order to control for noise due to model misspecification, the
30 joint estimates of our model were performed without weighting as required
31 by B&C (we will refer to this technique as a ‘direct’ estimation).

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33 The comparison among different models is performed through a simple
34 measurement, the squared Pearson correlation coefficient between observed
35 and fitted values. Results are proposed in Table 3. There are small differ-
36 ences between the B&C technique and the direct (unweighted) least squares
37 estimates (second and third column of Table 3). The joint model performs
38 much better than independent models (first column) for the first competi-
39 tor. Ranitidine’s trend, conversely, is also well described by an independent
40 model that essentially ignores competition.

Table 3: Squared Pearson correlation coefficient between observed and fitted values for alternative models.

ρ^2	INDEP. MODELS (BM)	CRCD MODEL (B&C)	CRCD MODEL (DIRECT)	KBKD MODEL (B&C)
CIMETIDINE (n=50)	0.994556	0.999051	0.999027	0.914248
RANITIDINE (n=40)	0.999757	0.999782	0.999788	0.991456
#parameters	6	8	8	6
\tilde{R}^2 w.r.t. CRCD	–	–	–	0.943905
F test	–	–	–	774.038

The KBKD model (fourth column) conversely performs poorly on these data (it is a special case of model (1) obtained when $p_2 = 0$ and $p_{1c} = p_{1a}$). With reference to the first constraint in our application, the innovative component of the second competitor was not negligible. Moreover, the choice of a common innovative parameter for the first competitor, both before and after the beginning of competition, is not supported by our data: in Table 2, we see that \hat{p}_{1a} significantly differs from \hat{p}_{1c} ; moreover, from Figure 4 we see that the fit of the KBKD model is much worse than the fit of our model. Since the KBKD model is nested within the CRCD model, we also calculated an F test to detect whether the gain from the simpler model to the more complex model was significant. In detail, as a first step, the squared multiple partial correlation coefficient

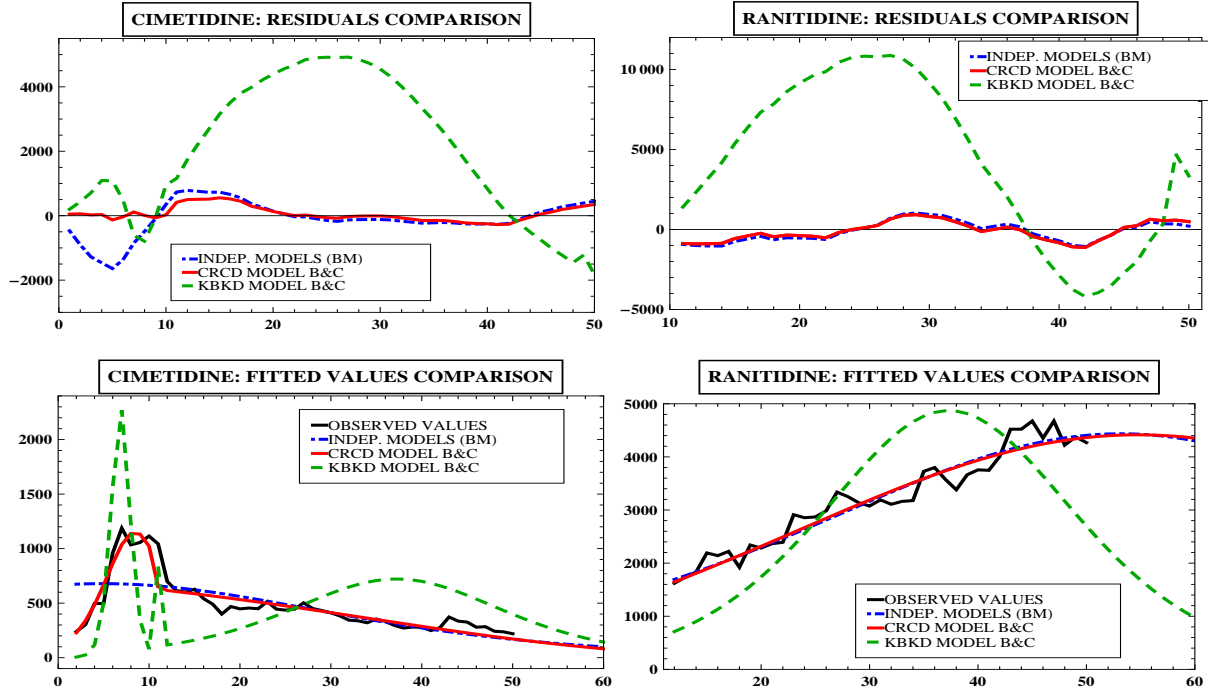
$$\tilde{R}^2 = (R_{M1}^2 - R_{M2}^2)/(1 - R_{M2}^2) \quad (13)$$

is calculated (here R_{M2}^2 denotes the determination index of the reduced model that has to be compared to model $M1$). A possible test to verify the significance of the s parameters of the $M1$ model that are not included in model $M2$ may be given by

$$F = [\tilde{R}^2(N - k)]/[(1 - \tilde{R}^2)s], \quad (14)$$

where N denotes the number of observations used to fit the models and k is

Figure 4: Comparisons among residuals (cumulative data) and rate fitted values of independent models, joint model CRCD, and KBKD model with the Beauchamp and Cornell (1966) technique.



the number of parameters included in model $M1$. Under the null hypothesis of equivalence between models $M1$ and $M2$, (14) is distributed as a Snedcor's F with $(s, N - k)$ degrees of freedom, if $\varepsilon(t)$ is normal i.i.d. Nevertheless, the F -ratio (14) can be used as an approximate robust criterion to compare model $M2$ nested in $M1$, by considering the well-known common threshold 4 (Guseo et al., 2007). Here the test comparing KBKD with CRCD assigns the huge value of 774.038, denoting the relevance of the extended model CRCD (see Table 3).

The analysis of residuals and fitted values (see Figure 4) confirms that the joint model (1), CRCD, is essential in order to capture the features of the first competitor while, for the second competitor, differences between the joint CRCD and independent models are, in this case, less appreciable. The KBKD is not performant in this situation.

Table 4: Squared Pearson correlation coefficient between observed and fitted values for alternative models.

ρ^2	CRCO MODEL (B&C)	LMPa MODEL Libai et al. (2009a) (B&C)	LMPb MODEL Libai et al. (2009b) (B&C)
CIMETIDINE (n=50)	0.999051	0.991492	0.994360
RANITIDINE (n=40)	0.999782	0.999568	0.998366
#parameters	8	5	6

We underline that in this application we decided not to include price and promotion variables, although the CRCO model can be easily extended to accommodate them, as explained in Appendix A. Prices are essentially neglected by physicians who prescribe patients such a drug (the cost is refunded by the National Health Service).

4.2. Comparison with some unbalanced models

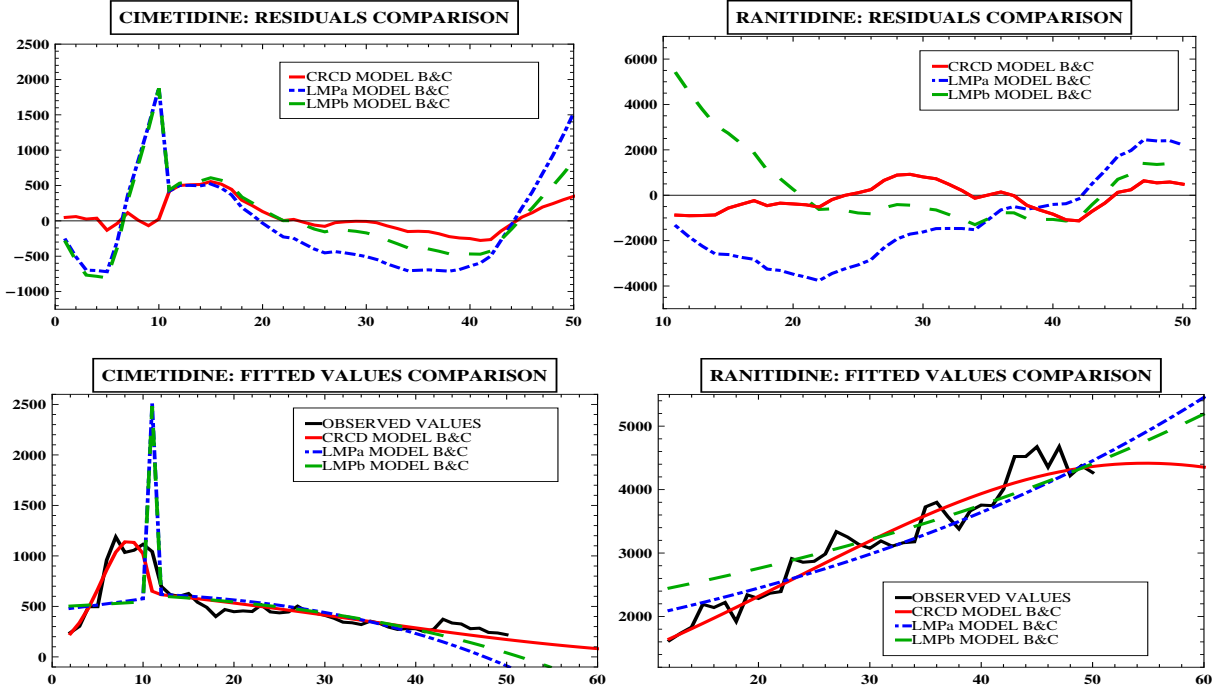
In order to compare our model with alternative structures, we fitted to our data the Libai et al. (2009a) model (LMPa) and the Libai et al. (2009b) model (LMPb). The aim was to see whether the balance assumption was too restrictive in our application. We show here the results in Table 4 and Figure 5. Our model performs better than the alternative ones.

5. Final remarks and discussion

Diffusion of innovation methodologies have faced and are facing new challenges in order to incorporate, in parsimonious model building, the major effects that can modify their evolutionary shapes over time.

The main result of this paper consists of the CRCO model. It considers a diachronic competition and extends the Krishnan et al. (2000) model, KBKD, in a natural way. We underline that the KBKD model was, within balanced

Figure 5: Comparisons among residuals (cumulative data) and rate fitted values of joint CRCD, LMPa, and LMPb models with the Beauchamp and Cornell (1966) technique.



models, the only known structure that provided a closed-form solution in the sequential entry situation.

We underline the essential role of a closed-form solution. The crucial question of the model choice can be supported by available data, through an efficient estimation method that exploits cumulative observations and avoids the cumbersome procedure needed to fit instantaneous data to the differential equations. Moreover, the analysis of the closed-form solutions allows a correct interpretation of the competition and substitution effects. This would not be possible through the qualitative study of the differential equations, because this method only describes the asymptotic behavior of the system and not the competition dynamics that have led to the equilibrium solution.

The closed-form solutions of CRCD highlight that the sales of each product are composed of a share of the whole category sales, corrected by two further departures. These take into account the role both of the level already

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7 reached by the first product when competition starts, and of the relative skills
8 in the diffusing of the two products along the whole life cycle. The relative
9 sizes of these components are evaluated through parameter estimates, and
10 their interpretation enables analysis of the interactions between the competi-
11 tors.
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15 The application to a case study concerning two competing active com-
16 pounds allows valuable interpretations regarding interactions induced by the
17 late entrant which, in this case, dominates the market but also delays the
18 declining behaviour of the first entrant through an expansion of the whole
19 category.
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23 We remind the reader that the CRCDC model can be easily extended to
24 deal with more than two competitors, and that closed-form solutions exist for
25 the most complex cases where the entrance of each new competitor generates
26 a regime change.
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30 The seminal paper by Bass (1969) has precipitated a wide set of contri-
31 butions, including the possibility of an external control effect through the
32 intervention function within the GBM framework (Bass et al., 1994). This
33 function allows a time domain control, expanding or reducing sales over time
34 under a fixed market potential. This useful re-allocation tool depends on
35 market-mix policies and strategic interventions. For the proposed twofold
36 CRCDC model, we have proposed a similar extension in Appendix A. Nev-
37 ertheless, further actions due to management policies, regulatory contexts,
38 and network externality effects may dynamically modify the market potential
39 (Guseo and Guidolin, 2009, 2010).
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47 In order to compare different models, we did not use some indicators
48 often introduced in the forecasting literature because this methodology is,
49 in our opinion, unconvincing. These indicators are based on the predictive
50 performance of models whose parameters are estimated through a shortened
51 time series (i.e., the final data are used only to measure predictions' accu-
52 racy). This approach, besides wasting the more informative data pertaining
53 to the more recent time periods, would be sensible only if we could assume
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7 ‘stationarity’ in the evolution uncovered by available data. This is a mean-
8 ingless assumption in real-world applications. For instance, a GBM and the
9 related extensions in Appendix A depend upon an *exogenous* intervention
10 function with an unpredictable evolution. For these reasons, we preferred to
11 analyse the complete data sets without restrictions to the right, and combine
12 global goodness-of-fit indexes with direct analysis of residuals with specific
13 reference to the right side (forecasting perspective).
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18 The main assumption of the present diachronic model CRCD is related
19 to the balanced word-of-mouth effects that do not take into account separate
20 influences induced by the within-brand relative knowledge as opposed to
21 the parallel cross-brand one. This assumption may be adequate for a wide
22 range of applications, where differences between brands are narrow. For
23 applications where that assumption is questionable, we refer to a further
24 contribution where the unbalanced version of this model will be analysed.
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30 As a final remark, we notice that the approach followed here emphasises
31 *aggregate* dynamic modelling, which ignores individual preferences and at-
32 titudes. In order to recognise different *local* diffusions, Cellular Automata
33 models (CA) and Network Automata (NA) are recent examples of the at-
34 tempt to take into account the heterogeneity of adopters. Boccara (2004),
35 among others, proposed interesting representations of special Cellular Au-
36 tomata models within the theory of Complex Systems. Nevertheless, such
37 models give rise to simulative frameworks that do not allow a stable and
38 well-characterised statistical inference. A much stronger argument is that, in
39 applied contexts, it is much more common to work with aggregate adoption
40 data that are cheaper, more reliable, and more often available. In Guseo and
41 Guidolin (2008, 2009), it has been proved, for a univariate case and under a
42 mean-field approximation, that there exist differential dual representations
43 of particular CA structures, driven by Riccati equations, that can be solved
44 in a closed-form. This property allows the use of well-founded statistical in-
45 ference in this nonlinear context. A special CA is proposed in Appendix C,
46 where an agent may select, at time t , at most one between two competing
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innovations. This *bivariate* structure can be simplified under a mean-field approximation, obtaining a continuous representation that gives rise to the synchronic version of the CRCDC model.

Appendix A. Competition and environmental intervention

The CRCDC model may be generalised if we introduce a common intervention function $x(t)$ following the ideas developed by Bass et al. (1994) for the Generalised Bass Model. Function $x(t)$ is positive definite and locally integrable. This function, which may depend upon exogenous variables, can modify the velocity of time elapsing within the *niche* that includes the two competitors. We underline here that we assume a common function $x(t)$ in order to express common dynamical properties of the competitive environment affecting both competitors. These properties may include only exogenous political or macroeconomic effects, as well as general effects due to expansion–competition interaction between the competitors.

If competition arises at time $c_2 > 0$ we have to follow the method designed in Section 3 with an updated model such that the existence of a common intervention function $x(t)$ is represented,

$$\begin{aligned}
 z'_1(t) &= m \left\{ \left[p_{1a} + q_{1a} \frac{z(t)}{m} \right] (1 - I_{t>c_2}) + \left[p_{1c} + q_{1c} \frac{z(t)}{m} \right] I_{t>c_2} \right\} \left[1 - \frac{z(t)}{m} \right] x(t) \\
 z'_2(t) &= m \left[p_2 + q_2 \frac{z(t)}{m} \right] \left[1 - \frac{z(t)}{m} \right] x(t) I_{t>c_2} \\
 m &= m_a (1 - I_{t>c_2}) + m_c I_{t>c_2} \\
 z(t) &= z_1(t) + z_2(t) I_{t>c_2}.
 \end{aligned} \tag{A.1}$$

If $t \leq c_2$, the generalised version of the aggregate (and first entrant) cumulative function is

$${}_{ag}z(t) = {}_{ag}z_1(t) = m_a \frac{1 - e^{-(p_{1a}+q_{1a}) \int_0^t x(\tau) d\tau}}{1 + \frac{q_{1a}}{p_{1a}} e^{-(p_{1a}+q_{1a}) \int_0^t x(\tau) d\tau}}, \quad t \leq c_2, \tag{A.2}$$

where we denote the final condition with ${}_g z_s = {}_{ag}z_1(c_2)$.

The aggregate cumulative function ${}_{cg}z(t)$ for $t > c_2$ is a Generalised Bass Model with initial condition ${}_gz_s$ for $t = c_2$ and has the following shape

$${}_{cg}z(t) = m_c \frac{1 - \left[\left(1 - \frac{{}_gz_s}{m_c}\right) / \left(1 + \frac{q}{p} \frac{{}_gz_s}{m_c}\right) \right] e^{-G(t)}}{1 + \frac{q}{p} \left[\left(1 - \frac{{}_gz_s}{m_c}\right) / \left(1 + \frac{q}{p} \frac{{}_gz_s}{m_c}\right) \right] e^{-G(t)}} I_{t \geq c_2}, \quad (\text{A.3})$$

with $G(t) = (p+q) \int_{c_2}^t x(\xi) d\xi$ and $p = p_{1c} + p_2$, $q = q_{1c} + q_2$. The new function, ${}_{cg}z_1(t)$, that depicts the behaviour of $z_1(t)$ for $t \geq c_2$ has a new form,

$${}_{cg}z_1(t) = {}_gH_1(t) - {}_gH_1(c_2) + {}_gz_s, \quad (\text{A.4})$$

where

$${}_gH_1(t) = \frac{m_c}{pf} \left[\left(p_{1c} - \frac{q_{1c}}{f} \right) \ln(f {}_gW + 1) + q_{1c} {}_gW \right] + K, \quad (\text{A.5})$$

with ${}_gE = \left[\left(1 - \frac{{}_gz_s}{m_c}\right) / \left(1 + \frac{q}{p} \frac{{}_gz_s}{m_c}\right) \right] e^{-(p+q) \int_{c_2}^t x(\tau) d\tau}$, ${}_gW = \frac{{}_{cg}z(t)}{m_c} = (1 - {}_gE)/(1 + f {}_gE)$ and $f = q/p$, so that equation (A.4) reduces to

$${}_{cg}z_1(t) = m_c \left[\frac{q_{1c}p - qp_{1c}}{q^2} \ln \left(\frac{f \frac{{}_gz_s}{m_c} + 1}{f {}_gW + 1} \right) + \frac{q_{1c}}{q} {}_gW \right] + \frac{q_2}{q} {}_gz_s. \quad (\text{A.6})$$

Analogously, for the second competitor we have

$${}_{cg}z_2(t) = m_c \left[\frac{q_2p - qp_2}{q^2} \ln \left(\frac{f \frac{{}_gz_s}{m_c} + 1}{f {}_gW + 1} \right) + \frac{q_2}{q} {}_gW \right] - \frac{q_2}{q} {}_gz_s, \quad t > c_2, \quad (\text{A.7})$$

where ${}_{cg}z_2(t) = 0$ for $t < c_2$. Under $x(t) = 1, t \geq 0$, ${}_gz_s = z_s$, ${}_gE = E$ and ${}_gW = W$. In that case, equations (A.6) and (A.7) reduce to (7) and (8).

In the simpler synchronic case, the solutions that generalise equations (12) under competition are

$$\begin{aligned} z_1(t) &= m \frac{q_1}{q} \frac{1 - e^{-(p+q) \int_0^t x(\tau) d\tau}}{1 + \frac{q}{p} e^{-(p+q) \int_0^t x(\tau) d\tau}} + m \frac{p}{q} \left(\frac{p_1}{p} - \frac{q_1}{q} \right) \ln \frac{1 + \frac{q}{p}}{1 + \frac{q}{p} e^{-(p+q) \int_0^t x(\tau) d\tau}} \\ z_2(t) &= m \frac{q_2}{q} \frac{1 - e^{-(p+q) \int_0^t x(\tau) d\tau}}{1 + \frac{q}{p} e^{-(p+q) \int_0^t x(\tau) d\tau}} + m \frac{p}{q} \left(\frac{p_2}{p} - \frac{q_2}{q} \right) \ln \frac{1 + \frac{q}{p}}{1 + \frac{q}{p} e^{-(p+q) \int_0^t x(\tau) d\tau}}. \end{aligned}$$

From the comparison of the previous solutions with system (12), it is easy to see how the function $x(t)$ affects the diffusion of the two brands.

Appendix B. Proof of equations (4) and (5)

We try to determine $z(t)$ within the two different regimes, namely $z(t) = {}_a z(t)(1 - I_{t > c_2}) + {}_c z(t)I_{t > c_2}$. Until $t = c_2$ we observe no competition, and then the global equation is the local one for $z_1(t)$,

$${}_a z'(t) = {}_a z'_1(t) = m_a \left[p_{1a} + q_{1a} \frac{z_1(t)}{m_a} \right] \left[1 - \frac{z_1(t)}{m_a} \right], \quad t \leq c_2, \quad (\text{B.1})$$

so that the aggregate solution, under initial condition ${}_a z(0) = {}_a z_1(0) = 0$, is

$${}_a z(t) = {}_a z_1(t) = z_1(t)I_{t \leq c_2} = m_a \frac{1 - e^{-(p_{1a} + q_{1a})t}}{1 + \frac{q_{1a}}{p_{1a}} e^{-(p_{1a} + q_{1a})t}}. \quad (\text{B.2})$$

The final cumulative condition at time $t = c_2$ is $z_s = {}_a z(c_2) = {}_a z_1(c_2)$. After c_2 , we observe competition, and the aggregate equation is

$${}_c z'(t) = z'(t) = m_c \left[p + q \frac{z(t)}{m_c} \right] \left[1 - \frac{z(t)}{m_c} \right], \quad t > c_2, \quad (\text{B.3})$$

with initial condition z_s at time $t = c_2$ and $p = p_{1c} + p_2$, $q = q_{1c} + q_2$. Equation (B.3) may be solved under previous condition (see, for instance, Bass, 1969, p. 218),

$${}_c z(t) = m_c \frac{1 + \frac{q}{p} \frac{z_s}{m_c} - \left(1 - \frac{z_s}{m_c} \right) e^{-(p+q)(t-c_2)}}{1 + \frac{q}{p} \frac{z_s}{m_c} + \frac{q}{p} \left(1 - \frac{z_s}{m_c} \right) e^{-(p+q)(t-c_2)}} I_{t > c_2} = m_c w(t) I_{t > c_2}. \quad (\text{B.4})$$

We now try to determine the $z_1(t)$ component under the competitive situation for $t > c_2$ and, coherently with the previous notation, we denote such a solution with a special subscript c , ${}_c z_1(t) = z_1(t)I_{t \geq c_2}$.

We consider, preliminarily, a new position,

$$E = \left[\left(1 - \frac{z_s}{m_c} \right) / \left(1 + \frac{q}{p} \frac{z_s}{m_c} \right) \right] e^{-(p+q)(t-c_2)}. \quad (\text{B.5})$$

Equation (B.4) is equivalent, for $f = q/p$, to

$$W = w(t) = \frac{{}_c z(t)}{m_c} = \frac{1 - E}{1 + fE} \quad (\text{B.6})$$

and, in particular, we observe $w(c_2) = z_s/m_c$. Conversely, the expression of t as a function of W is equal to:

$$t = \frac{1}{p+q} \ln \left(\frac{1 + \frac{q}{p}W}{1-W} \right) + c_2 + \frac{1}{p+q} \ln \left(\frac{1 - \frac{z_s}{m_c}}{1 + \frac{q}{p} \frac{z_s}{m_c}} \right)$$

and $dt = [p(1-W)(1 + \frac{q}{p}W)]^{-1}dW$. Since we are considering $t > c_2$, integration of the first equation in (1) gives rise to

$${}_c z_1(t) = \frac{m_c}{p} \int \frac{p_{1c} + q_{1c}W}{1+fW} dW, \quad (\text{B.7})$$

and its general solution is

$$H_1(t) = \frac{m_c}{pf} \left[\left(p_{1c} - \frac{q_{1c}}{f} \right) \ln(fW + 1) + q_{1c}W \right] + C, \quad (\text{B.8})$$

with C a generic undetermined constant. We may compute the definite integral within the range $[w(c_2), W]$ and, therefore, we obtain

$$\begin{aligned} {}_c z_1(t) &= H_1(t) - H_1(c_2) + z_s \\ &= \frac{m_c}{pf} \left\{ \left(p_{1c} - \frac{q_{1c}}{f} \right) \ln \left[\frac{fW + 1}{fW(c_2) + 1} \right] + q_{1c}[W - W(c_2)] \right\} + z_s \\ &= m_c \left[\frac{qp_{1c} - q_{1c}p}{q^2} \ln \left(\frac{fW + 1}{f \frac{z_s}{m_c} + 1} \right) + \frac{q_{1c}}{q} W \right] + \frac{q_2}{q} z_s. \end{aligned} \quad (\text{B.9})$$

Notice that, for $c_2 = 0$, we have $z_s = {}_a z_1(c_2) = m_c w(c_2) = 0$, and then we attain the synchronic solution (12) by backward substitution.

In order to write equation (B.9) as an explicit function of t , we further develop it, starting from the argument of the logarithm:

$$\frac{fW + 1}{f \frac{z_s}{m_c} + 1} = \frac{1 + \frac{q}{p}}{1 + \frac{q}{p} \frac{z_s}{m_c} + \frac{q}{p} \left(1 - \frac{z_s}{m_c} \right) e^{-(p+q)(t-c_2)}} = y(t). \quad (\text{B.10})$$

Substituting expression (B.10) into (B.9) gives, after simple simplifications, expression (7). The profile of the second or late entrant is very simple. For $t \leq c_2$ we have ${}_a z_2(t) = {}_a z(t) - {}_a z_1(t) = 0$. For $t > c_2$ we obtain ${}_c z_2(t) = {}_c z(t) - {}_c z_1(t)$ or, equivalently,

$$\begin{aligned} {}_c z_2(t) &= m_c w(t) - m_c \left[\frac{q_{1c}}{q} w(t) + \frac{p}{q} \left(\frac{p_{1c}}{p} - \frac{q_{1c}}{q} \right) \ln y(t) \right] - \frac{q_2}{q} z_s \\ &= m_c \frac{q_2}{q} w(t) - \frac{q_2}{q} z_s + m_c \frac{p}{q} \left(\frac{p_2}{p} - \frac{q_2}{q} \right) \ln y(t). \quad \square \end{aligned}$$

Appendix C. Multivariate Cellular Automata

The model proposed in this paper was built focusing on the evolution of the aggregate sales of a *population* of individuals considered as a unique entity. We will show here that the synchronic version of the CRCDC model could be motivated also in a very different way, since it represents an approximation of a multivariate Cellular Automata (CA) system, describing the behaviour of *single agents*. The key issue of this connection can be explained as follows. The stochastic rule (C.4) below defines a simple evolution of a CA and may be simulated on the basis of a specification of the involved parameters p, q, w and m . A more interesting problem is obviously the estimation of these unknown parameters with available data. Usually, *individual* data are not available at time t , due to privacy constraints or costs, while *aggregate* information about the general state of the system may be easier to handle. For this reason the dualism between an individual level system and an aggregate level model allows a switch between the two representations according to the aims.

A deterministic univariate Cellular Automaton (CA) is characterised by three elements: a *population of agents* (cells), Z ; a *state function*, $s(i, t)$; and a *local evolutionary rule*, $f(\cdot)$. The population of agents, Z , has a one-to-one correspondence with a set of labels for agents' identification. We assume Z is the set of all integers. According to Boccara and Fuks (1998), the state function $s(i, t) \in Q$ denotes, for each agent $i \in Z$ at time $t \in \mathbb{N}^*$ (the set of all positive integers), a level within the class $Q = \{0, 1\}$ of possible states: $s(i; t) = 1$ denotes the adoption of a particular innovation by agent i and, conversely, $s(i; t) = 0$ depicts the neutral state. The local evolutionary rule (transition rule) is a function $f : Q^{2r+1} \rightarrow Q$, such that

$$s(i; t + 1) = f(s(i - r; t), s(i - r + 1; t), \dots, s(i - 1 + r; t), s(i + r; t)), \text{(C.1)}$$

where the integer r , is the *radius* of the rule.

Let us define for $i \in Z$ a kind of *local pressure* of the system, $0 \leq \sigma_s(i; t) \leq 1$, depending on a flexible probability measure, $p_n \geq 0$, that allows a more

general description of a neighboring stimulating effect towards adoption:

$$\sigma_s(i; t) = \sum_{n=-\infty}^{\infty} s(i+n; t)p_n; \quad \sum_n p_n = 1. \quad (\text{C.2})$$

If local pressure is *translational invariant*, we may consider the mean-field approximation. This reduction excludes the local effect of distribution p_n ,

$$\sigma_s(i; t) = \lim_{r \rightarrow \infty} \sum_{j=-r}^r \frac{s(i+j; t)}{2r+1} \simeq \nu(t) = \frac{z(t)}{m}, \quad (\text{C.3})$$

where $\nu(t)$ depicts the ‘density’ of the adoption process or the normalised ratio $\nu(t) = z(t)/m$ with m the assumed constant market potential and $z(t)$ the cumulative product sales at time t .

Let us define a special rule $f(\cdot)$ under $Q = \{0, 1\}$ through a partially probabilistic specification,

$$\begin{aligned} s(i; t+1) &= \\ &= s(i; t) + [Bi(1, p) \oplus Bi(1, q \sigma_s(i; t))] I_{(s(i; t)=0)} - Bi(1, w) I_{(s(i; t)=1)} \\ &= s(i; t) + Bi(1, p + q \sigma_s(i; t)) I_{(s(i; t)=0)} - Bi(1, w) I_{(s(i; t)=1)}, \end{aligned} \quad (\text{C.4})$$

where \oplus denotes a *selection rule* between two *mutually exclusive* components. The first innovative component of equation (C.4), $Bi(1, p)$, depends upon a binomial experiment, with parameter p , which is realisable only if the indicator function $I_{(s(i; t)=0)}$ is set to one, that is, proposition $(s(i; t) = 0)$ is true. The meaning of such a first component may be linked to the effect of mass media communication channels. The change of state is possible, with probability p , only if such ‘institutional communication’ reaches the susceptible agent i , which supports the initialising aspects of an adoption process. We consider such an agent to be an *innovator*. The second component of equation (C.4), $Bi(1, q \sigma_s(i; t))$, considers the joint probability $q \sigma_s(i; t)$ that depicts the local pressure effect of a neighboring social practice to adopt, $\sigma_s(i; t)$, combined with the intrinsic attitude to pure imitative response pushed by a parameter q . This second experiment is an opportunity strictly referred to standard agents and expresses the commonly perceived fact that imitative behaviour

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7 is twofold: an individual attitude combined with a local pressure due to the
8 neighboring environment. We consider such an agent to be an *imitator*. No-
9 tice that activation of the previous two components is strictly alternative. In
10 other words, under condition $I_{(s(i;t)=0)} = 1$, an agent selects only one bino-
11 mial experiment, if any, at his free choice, so that the resulting framework is
12 a special binomial experiment for the external observer, $Bi(1, p + q\sigma_s(i; t))$.
13 In particular, the binomial parameter $p + q\sigma_s(i; t)$ represents the marginal
14 adoption probability (mixture) based on the averaging of the corresponding
15 group conditional adoption probabilities, 1 for innovators, $\sigma_s(i; t)$ for imita-
16 tors and zero for neutral agents. The group weights are p , q and $1 - p - q$,
17 respectively, so that we attain $p \cdot 1 + q \cdot \sigma_s(i; t) + (1 - p - q) \cdot 0 = p + q\sigma_s(i; t)$. As
18 mentioned above, we denote, with the term *selection rule*, such a composition
19 rule. Finally, the third component in equation (C.4) is a decay effect driven
20 by a binomial $Bi(1, w)$ under the control of the correct state, $I_{(s(i;t)=1)}$, and
21 describing a possible withdrawal from an active state.
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32 Let us consider, therefore, the average behaviour of rule (C.4), under a
33 mean-field approximation expressed by equation (C.3), followed by the sum
34 of all states indicators $s(i; t)$ within Z divided by $2r + 1$. If the limit exists,
35 we have
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$$\nu(t + 1) = \nu(t) + (p + q\nu(t))(1 - \nu(t)) - w\nu(t). \quad (\text{C.5})$$

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40 We can approximate the discrete time equation (C.5) with a continuous
41 Riccati equation, namely,
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$$\nu'(t) = -q\nu^2(t) + (q - p - w)\nu(t) + p, \quad (\text{C.6})$$

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47 and, if we exclude the exit rule component, $w = 0$, we have a standard Bass
48 (1969) model. Solution $\nu(t)$ of equation (C.6) is described in Guseo and
49 Guidolin (2008). We underline that $z(t) = m\nu(t)$ defines an absolute aggre-
50 gate temporal evolution of the proposed Cellular Automaton.
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56 The previous approach can be extended to a competitive market. We
57 represent an automaton where an agent at time t may remain neutral, or
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select, without loss of generality, at most only one between two competing innovations. As we did before, we define with an indicator function $I_{(s(i;t)=0)} = I_{(s_1(i;t)+s_2(i;t)=0)}$ the condition under which agent i has not performed any adoption at time t , where $s_1(i;t)$ and $s_2(i;t)$ denote the state functions related to two different innovations, namely, innovation 1 and innovation 2. Their sum, $s(i;t) = s_1(i;t) + s_2(i;t)$, is at most one. If we exclude the exit rule component, $w = 0$, we may generalise equation (C.4), simultaneously representing the transitions rules for both state functions, as follows:

$$\begin{aligned} s_1(i;t+1) &= s_1(i;t) + Bi(1, p_1 + q_1\sigma_s(i;t))I_{(s(i;t)=0)} \\ s_2(i;t+1) &= s_2(i;t) + Bi(1, p_2 + q_2\sigma_s(i;t))I_{(s(i;t)=0)}, \end{aligned} \quad (\text{C.7})$$

where $\sigma_s(i;t) = \sum_{n=-\infty}^{\infty} s(i+n;t)p(n)$ and $\sum_n p_n = 1$. In particular, $\sigma_s(i;t)$ is a common pressure towards adoption, based on the knowledge of the product category; it does not depend on the specific brand/product.

Note that if we sum previous synchronic processes in equations (C.7) we obtain a category transition rule,

$$s(i;t+1) = s(i;t) + Bi(1, p + q\sigma_s(i;t))I_{(s(i;t)=0)}, \quad (\text{C.8})$$

where $p = p_1 + p_2$ and $q = q_1 + q_2$. The reason for this result is that the sum of the binomial experiments in equations (C.7) follows the *selection rule* (the agent selects at most *one* brand). This explains, with different terminology, the source of competition.

We may approximate the aggregate discrete time system (C.7) with a continuous representation, under the mean-field approximation described in equation (C.3),

$$z'_1(t) \simeq \sum_i [s_1(i;t+1) - s_1(i;t)] \quad \text{and} \quad z'_2(t) \simeq \sum_i [s_2(i;t+1) - s_2(i;t)],$$

where, in particular, $z(t) = \sum_i s(i;t) = \sum_i [s_1(i;t) + s_2(i;t)] = z_1(t) + z_2(t)$. These positions give rise to the differential counterpart, the synchronic version of the CRCDC model described in Section 3, and confirm analogous results

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7 expressed in Guseo and Guidolin (2008, 2009) that establish a dualism be-
8 tween Complex Systems' representations based on Cellular Automata and
9 the corresponding mean-field aggregate versions based on traditional differ-
10 ential equations systems.
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