Diffusion of technological innovations: dynamic and static equilibria

Renato Guseo

Abstract The catenary function has a well-known role in determining the shape of chains and cables supported at their ends under the force of gravity. This enables design using a specific static equilibrium. Its symmetric version, the catenary arch, allows the construction of bridges and arches exploiting the dual equilibrium property under uniform compression. In this paper we emphasise a further connection with well-known biological growth models and the related diffusion of innovation paradigms, e.g., logistic and Bass functions, that determine self-sustaining evolutionary growth dynamics in naturalistic and socio-economic contexts.

Key words: hyperbolic cosine, catenary function, logistic model, Bass model.

1 The Catenary and some diffusion models: An introduction

It is sometimes surprising to discover how certain natural and social phenomena may refer to common interpretations in a way that is universal.

Here we propose a careful consideration of a typical function of mathematical analysis, the catenary. This function, referable to the hyperbolic cosine, directly describes such different phenomena as: ropes or chains supported at both ends and subjected to the action of gravity, the shape of the cables of suspended power lines. Its simple transformation, reflection, describes arches in many buildings, and the supporting arches of bridges. A further simple transformation, based essentially on the square root of the reciprocal, represents the logistic distribution head of a family of models that describes and predicts the growth of biological systems, but also the diffusion of ideas and knowledge in a social body, as well as the market penetration of specific technological innovations or fashions.

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The function that defines the *catenary* in Cartesian coordinates takes the form,

$$y = a\cosh\left(\frac{x}{a}\right) = \frac{a}{2}\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right), \tag{1}$$

where $\cosh(\cdot)$ is called hyperbolic cosine. Its derivation is omitted for brevity. A reflected version of the catenary solves the dual equilibrium problem, i.e., the construction of a freestanding arch with uniform loads.

The term used for the catenary (1) comes from the Latin word *catena* and was introduced by Huygens in a letter to Leibniz in 1690. Galilei also examined this curve. In the *Dialogue of the Fourth Day* he corrects his previous sentence by noting that the approximation to the parabola is obtained in the presence of a small curvature.

The use of the catenary arch in architecture, as an ideal self-sustaining arch in relation to uniform loads, draws upon the premises of Hooke, but its actual use is much more recent. Across Europe, various architectural movements arise simultaneously at the turn of the nineteenth century and the twentieth century. The *Art Nouveau* in France, the *Liberty* in Italy, and the *Jungendstil* in Germany and Austria are flanked by the so-called *Catalan Modernism*. The outstanding creator of this current of thought is Antoni Gaudì (1852 - 1926). He built several buildings including a dozen which are defined by UNESCO as World Heritage Sites, among them: Park Güell, Palau Güell, Casa Batllò, Casa Milà, Casa Vincens, the crypt of La Colonia Güell, and La Sagrada Familia. For Gaudì the catenary is an organic component of architectural buildings: it ensures a static equilibrium.

The phenomena of growth are found in many natural and social contexts. These systems consist of a large number of agents that are connected in terms of relational forms through appropriate languages; see, for instance, Couzin [4], and Pentland [11]. An example is a swarm of bees. The organisation of these systems does not require a central control. For example, the initiatives linked to the search for food are based on flexible recruitment of worker bees. Based on preliminary information provided by some explorer bees through conventional dances, many other workers take steps to reach the site, dynamically involving other bees. The dynamics of the spread of a viral agent in a human population is a function of types of organisation and networks of social relationships (Barabasi, [1]). The launch of a new product or a fad, responds to similar mechanisms (Rogers [12]; Granovetter [5]; Bass [2]; Bass et al. [3]). Some opinion leaders quickly adopt innovation based on the actions of corporate communications; then the mechanism is activated in parallel by word of mouth, a very powerful tool that is critical to the success or the failure of almost all business initiatives. More recent and advanced results in diffusion of innovations, based on Cellular Automata representations, may be found in Guseo and Guidolin [6, 7, 8, 9], and in Guseo and Mortarino [8]. The basic case studies are represented by the logistic model (Verhulst [13]) and the Bass model [2].

The logistic equation expresses the connection between the instantaneous change of a phenomenon $z'(t)$, the cumulative extension of the current process, $z(t)$, and the residue which may still be activated, $(m - z(t))$. The equation that governs the dynamics can take the following form, $z'(t) = rz(t)(m - z(t))/m, t \in R$, where the parameter $r > 0$ controls the speed of the reaction. The above equation is completed with an initial *positive* condition $z(0) = z_0 > 0$. Under the position $t_p = \frac{1}{r}\log\frac{m-z_0}{z_0}$, the solution is $z(t) = \frac{m}{1+e^{-r(t-t_p)}} = mL(t)$, where $L(t)$ represents the distribution function of the corresponding logistic distribution. The instantaneous growth rate is

$$z'(t) = \frac{mre^{-r(t-t_p)}}{\left(1+e^{-r(t-t_p)}\right)^2} = ml(t). \tag{2}$$

It can be shown that t_p is the time to peak and it is obviously connected with the initial condition: $z_0 = m/(1+e^{rt_p})$. The prevailing applications of the logistic growth model are usually defined within naturalistic fields.

The Bass equation [2] has established itself in a very different context with respect to the former, in the so-called processes of diffusion of innovations (Rogers [12]). This is a typical field of applied mathematics at the quantitative marketing. Bass suggests that success in marketing a product or service depends very much on the two main types of consumers: innovators who are directly sensitive to the actions of institutional communication, and imitators who basically ignore this channel of information and prefer to support their decisions on the basis of interpersonal relationships. In the latter case, the main channel is the *word of mouth* in a broad sense including the signs and gestures that humans used broadly (Pentland [11]). The proposed equation takes the following form, $z'(t) = (p+qz(t)/m)(m-z(t)), t \in [0,+\infty)$, where the parameter $p > 0$ represents the contribution of innovators directly proportional to the residual market $(m-z(t))$ and $q > 0$ rules, so modulated by the relative knowledge about the product, $z(t)/m$, the access to the residual market $(m-z(t))$ due to the imitators. The above equation is completed with an initial condition which is natural for processes of this type, $z(0) = 0$. The solution of the cumulative Bass model is, $z(t) = m(1-e^{-(p+q)t})/(1+(q/p)e^{-(p+q)t}), t \in [0,+\infty), 0 < p < q$, and the correspondig rate version is

$$z'(t) = \frac{m(p+q)^2 e^{-(p+q)t}}{p\left(1+\frac{q}{p}e^{-(p+q)t}\right)^2}, \quad t \in [0,+\infty), \quad 0 < p < q. \tag{3}$$

We can compare the logistic model with that of Bass using a reparameterisation of equation (3), i.e., $r = p+q$, $t_p = (\ln q/p)/(p+q)$.

We obtain, therefore, $z(t) = mB(t) = m(1-e^{-rt})/(1+e^{-r(t-t_p)}), t \in [0,+\infty), 0 < r$, where $B(t)$ is the Bass distribution function, and

$$z'(t) = mb(t) = \frac{mr\left(e^{-rt}+e^{-r(t-t_p)}\right)}{\left(1+e^{-r(t-t_p)}\right)^2}, \quad t \in [0,+\infty), \quad 0 < r, \tag{4}$$

the corresponding rate function with $b(t)$ the Bass density.

Note that the cumulative Bass model can be determined by a monotonic transformation of the logistic model (isomorphism): $B(t) = L(t)(1-e^{-rt}), t \in [0,+\infty)$. In the logistic model the mechanism of initialisation is entirely concentrated at the time $t=0$ by the positive initial condition. In the Bass model initialisation is based on a ‘seeding effect’ distributed over time.

2 The Catenary and some diffusion models: A relationship

For convenience, we examine here the connection of a perturbed catenary with the corresponding perturbed logistic model, knowing that this moves, in accordance with the isomorphism, as the Generalised Bass model [3].

A logistic equation perturbed by an intervention function $x(t)$ is $z'(t)=\frac{r}{m}z(t)(m-z(t))x(t), t\in R$. Its solution, under initial condition $z(0)=z_0>0$ or, $t_p=\frac{1}{r}\log\frac{m-z_0}{z_0}$, is $z'(t)=mf(t)=rx(t)e^{-r\left(\int_0^t x(\tau)d\tau-t_p\right)}/\left(1+e^{-r\left(\int_0^t x(\tau)d\tau-t_p\right)}\right)^2$. The corresponding perturbed catenary, for $\sigma=2/r$, is

$$\frac{\sqrt{\frac{\sigma}{2}}}{\sqrt{f(t)}}=\frac{1}{\sqrt{x(t)}}\sigma\cosh\left(\frac{\int_0^t x(\tau)d\tau-t_p}{\sigma}\right). \tag{5}$$

There is a strong connection between a static equilibrium in a freestanding arch and the corresponding dynamic equilibrium of a related diffusion process. For $x(t)=1$ we obtain the pure logistic density and the related catenary.

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