Modelling a Dynamic Market Potential:

A Class of Automata Networks for Diffusion of Innovations

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Abstract

Innovation diffusion processes are generally described at aggregate level with models like the Bass Model (BM) and the Generalized Bass Model (GBM). However, the recognized importance of communication channels between agents has recently suggested the use of agent-based models, like Cellular Automata. We argue that an adoption or purchase process is nested in a communication network that evolves dynamically and indirectly generates a latent non-constant market potential affecting the adoption phase.

Using Cellular Automata we propose a two-stage model of an innovation diffusion process. First we describe a communication network, an Automata Network, necessary for the “awareness” of an innovation. Then, we model a nested process depicting the proper purchase dynamics. Through a *mean field approximation* we propose a continuous representation of the discrete time equations derived by our nested two-stage model. This constitutes a special non-autonomous Riccati equation, not yet described in well-known international catalogues. The main results refer to the closed form solution that includes a general dynamic market potential and to the corresponding statistical analysis for identification and inference. We discuss an application to the diffusion of a new pharmaceutical drug.

Keywords: Diffusion process; Bass model; Communication network; Cellular Automata; Riccati equation; New drugs.

1. Introduction

Since the publication of the Bass model in 1969, research on diffusion of innovations and innovation theory have raised a growing interest, with reference both to consumers behaviour (see Gatignon and Robertson [1]) and marketing management for developing new strategies focused on potential adopters or potential adoption units. Interesting reviews of the literature on diffusion models are provided by Mahajan and Muller [2]), Mahajan *et al.* [3], Mahajan *et al.* [4], Meade and Islam [5] and Muller *et al.* [6] where it is highlighted that the purpose of the diffusion model is to describe the successive increases in the number of adoptions or purchases and predict the continued development of a diffusion process already in progress. In spite of the more recent research proliferation in this field, the basic and fundamental known diffusion models are those of Fourt and Woodlock [7], Mansfield [8] and Bass [9]. The last one (BM) is an extension of the other two and assumes that potential adopters (or adoption units) are influenced in their purchase behaviour by two sources of information: an external, like mass-media communication and an internal, word-of-mouth. Furthermore, it is assumed that adopters can be influenced only by one of these two forces, forming two distinct groups, innovators (mass-media) and imitators (word-of-mouth) and therefore, part of adoptions is based on learning by imitation and part of them does not. Formally, the model can be expressed through a first order differential equation

$$z'(t) = \left(p + q\frac{z(t)}{m} \right)(m - z(t)) \tag{1}$$

Instantaneous purchases, $z'(t)$, at time t are proportional to the residual market $(m - z(t))$, where m is the assumed fixed market potential, and determined by two additive components. The first one, $p(m - z(t))$ refers to innovators, who adopt with a rate p called coefficient of innovation. The group of innovators is essential for the "take-off'" of diffusion, even if present at any stage of the process. The second part of Equation (1), $q(z(t)/m)(m - z(t))$ represents purchases of buyers at time t who are influenced by previous realized adoptions (word-of-mouth effect, w-o-m for short) through parameter q. The effect of parameter q is modulated by the ratio $z(t)/m$, justifying the temporal delay of purchases due to w-o-m effect. In what follows we assume that w-o-m describes both direct interpersonal communication including "auto-communication" (learning and memory effects) in different times and signals due to realized adoptions.

If innovators are necessary for the initial phase of the diffusion process, imitators are crucial for its development and growth, the life cycle of an innovation depending on these two combined effects.

An equivalent interpretation of Equation (1) is based on the dual hazard rate specification, i.e., $h(t) = z'(t)/(m - z(t)) = p \cdot 1 + q \cdot z(t)/m + (1 - p - q) \cdot 0$, where p, q and $1 - p - q$ are group conditioning probabilities with $1-p-q$ pertaining to "neutrals" subgroup and 1, $z(t)/m$ and 0 are the corresponding conditional probabilities towards adoption. Summation over previous joint probabilities describes the usual marginalization technique based on the law of total probability.

An extremely useful extension of the Bass model is represented by the Generalized Bass Model (GBM) by Bass *et al.* [10] allowing to include the presence of exogenous interventions (strategic interventions, policies, marketing strategies). The GBM equation is

$$z'(t) = (p + q\frac{z(t)}{m})(m - z(t))x(t) \quad (2)$$

where $x(t)$ denotes a quite general intervention function with neutral level 1, whose effect can accelerate or delay purchases over time, i.e., control the geometry of time as we can see by inspecting, for instance, the closed form solution of Equation (2) in [10]. Function $x(t)$ cannot control the market potential m or the intrinsic diffusion parameters p and q.

One of the main assumptions in the Bass models relates to the market potential (or carrying capacity), m, whose size is considered fixed along the whole diffusion process. However, it is a common experience that such a potential may have a variable structure during the product life cycle, due to endogenous or exogenous factors.

The issue of a dynamic market potential is not new to the diffusion literature. We may separate two different formal approaches: Some papers introduce only a modification of the residual market, $(m - z(t))$ and exclude an intervention on the w-o-m ratio. See, for instance, Mahajan and Peterson [11], Horsky [12], Kamakura and Balasubramanian [13] and Mesak and Darat [14].
A second group allows for both modifications. Among others, we consider Sharif and Ramanathan [15], Jain and Rao [16], Parker [17], [18], Rao [19], Kim *et al.* [20], Goldenberg *et al.* [21] and Centrone *et al.* [22].

From a mathematical point of view there are different assumptions governing the shape of $m(t)$. In some cases it is exogenously determined as a function of observed variables, e.g., in Mahajan and Peterson [11], Kalish [23], Jain and Rao [16], Parker [17], [18], Horsky [12], Kim *et al.* [20] and Kamakura and Balasubramanian [13]. Some effort is necessary for a correct specification of the main drivers (population, prices, number of households with special facilities, number of competitors, number of retailers, threshold probabilities, etc.) and a suitable transformation to attain a reasonable correspondence with the scale of the adoption process.

In few cases the market potential is assumed to follow a simple exponential function of time, Sharif and Ramanathan [15], Centrone *et al.* [22] and Meyer and Ausubel [24].

In this paper we propose a model in which the market potential is a function of time, $m(t)$, and may assume various levels during the product life cycle. We observe that such a variability is particularly evident in the first part of diffusion, the so called *incubation period*, when the success of an innovation is still uncertain and may depend on several elements, eventually depressing sales. We argue that in this phase marketing and management activities play a crucial role in stimulating the product "take off".

However, there are situations in which these efforts are not sufficient for overcoming the initial crisis: a typical example is represented by those goods that produce network externalities effects due to technological constraints (mobile phones, fax-machines, etc.). Thus we focus our research on goods that do not exhibit network externalities effects. We concentrate our interest on "stand alone goods" whose adoption may be highly facilitated by promotional activities and, in particular, we examine a new pharmaceutical drug's diffusion.

As is well-known, advertising and other forms of product promotion exert a major effect in the launch phase, while other kinds of communication like word-of-mouth may better explain later purchases.

We use this consideration for motivating the time dependence of the market potential and we adopt an evolutionary perspective for providing a theoretical explanation of this intuition. In particular, we develop a model in which communication and adoption processes are separate but co-evolutionary phases in diffusion.

The main idea expressed in our model may find a precursory version in Sawhney and Eliashberg [25]. The Authors propose a two-phase approach by separating two notions of time: the *time to decide, T,* and the *time to act,* τ. The time to adopt is the sum of the previous two, $t = T + \tau$, and its distribution is a convolution under very simple exponential frameworks. However, our approach is different, being based on the direct product of two independent distributions that define the normalized market potential (awareness-persuasion) and the corresponding normalized purchase process.

We use the term awareness to indicate also persuasion to act: not a simple knowledge, but "knowledge under way".

The concrete implication of our model, is that we are able to estimate, under appropriate theoretical assumptions, the evolving structure of a *latent* market potential simply using cumulative *sales data*.

This is of particular concrete interest because it allows to measure indirectly the receptiveness of a social context, facilitating comparisons between different situations and evaluations on the effectiveness of firms' marketing efforts.

The paper is organized as follows. In Section 2 we present a plausible rationale for a dynamic market potential definition based on an extended formal interpretation of absorptive capacity by Cohen and Levinthal [26] within Complex Systems. In Section 3 we model a stochastic evolution of a communication network, generalizing the binary Automata Network proposed by Boccara *et al.* [27] and we define accordingly a dynamic market potential $m(t)$. In Section 4 we present a Cellular Automata (CA) model for a co-evolutionary adoption or purchase process whose *mean field approximation* yields a Riccati equation.

The closed form solution of a general non-autonomous Riccati equation, which, under particular constraints, provides the standard Bass model and the Generalized Bass Model (GBM), is proposed in Appendix A. In Section 5 we apply previous results to the co-evolutionary model and examine statistical aspects related to inference and applications. Section 6 analyzes an application to the diffusion of a new pharmaceutical drug. Final comments and discussion are considered in Section 7.

2. Absorptive Capacity in a Complex System: a Rationale for a Dynamic Market Potential

As we have seen, the Bass model proposes a simple and efficient bipartition of consumers' behaviour based on information channels. Of course we are not saying anything new if we point out the relevance of information for economic action. However, starting from a basic level of reasoning, according to which a consumer adopts after being informed about an innovation (its existence and its features), we could investigate more in detail the relationship between information and innovation diffusion.

A relevant contribution, more in qualitative terms, on this topic has been given by Cohen and Levinthal [26] that defined the concept of *absorptive capacity*. Even though the authors' focus is the firm, we think that very similar considerations may be easily applied in a consumption perspective.

Considered both at the individual and organizational level, the term absorptive capacity refers to the “ability to recognize the value of new information, assimilate it and apply it”.

In our view, it is an extended notion of awareness that incorporates persuasion to act or adopt. It is argued by the authors that this ability to assimilate and exploit a novelty is function of a *prior related knowledge*. That is, the presence of a background of relevant knowledge implies a greater *receptiveness* to new ideas. Cohen and Levinthal use this concept both for individuals and systems.

As they point out, in the individual case, this ability is related to cognitive functions of the single person, while to understand a system's absorptive capacity it is necessary to focus on its communication structure, since this capacity is not the simple sum of those of its components, but has to do with knowledge transfers.

The concept of absorptive capacity in social systems is particularly interesting for the purposes of this paper, in which we focus on innovation phenomena at the aggregate level.

The adoption of an innovation in a specific social context may be viewed as a direct evidence of an existing absorptive capacity: in fact, the ability to assimilate and accept a novelty may find a simple check in the observed purchase process. Specifically, the market potential m may represent a measure of this absorptive capacity.

Since the ability to assimilate an innovation depends on the *accumulation of a prior knowledge*, we could try to define the market potential accordingly. A process of accumulation of knowledge in a social system requires the transfer of information among the components of the system. In this sense Cohen and Levinthal highlight the importance of designing the communication structure of an organization to understand its absorptive capacity. Accumulating knowledge involves some learning dynamics, whose description, in our view, is best reached through an evolutionary model, rather than a cross-sectional modelling, as proposed by Cohen and Levinthal [26].

Developing an *evolutionary perspective*, we represent a communication structure as a set of informational linkages among the units of the system. As individual knowledge is created connecting ideas and concepts between them but also destroying some existing connections, the development of a *collective knowledge* can be thought of as an *evolving network*, in which some linkages exist, some rise and some others die. The existence and evolution of this network allow to realize that a collective knowledge is based essentially on social consensus.

Considering the market potential $m(t)$ as a function of this knowledge process, will imply to make it dependent on a network of connections that changes over time.

Recent studies (see for instance, Mahajan *et al.* [28]; Eliashberg *et al.* [29]; Muller *et al.* [6]) have confirmed that internal communication forces play a key role in new product diffusion. However, little is known on how this interpersonal communication is structured.

Though innovation diffusion theory has been defined as a theory of communications, the Bass model is rather silent in exploring the effect of the communication process underlying adoptions.

If the Bass model is generally able to capture the macro-behaviour of diffusions through three parameters (m, p, q), the analysis of the underlying micro-interactions is left to other kinds of models (see, for instance, Chatterjee and Eliashberg [30] and Roberts and Lattin [31]) or methods dealing with the issue of *complexity*. See, in particular, a stimulating review in Muller *et al.* [6]. Goldenberg *et al.* [32] suggest that the gap of knowledge on communication may be linked to the complexity of this process, which may be described as a “complex adaptive system”, i.e., a system consisting of many interacting agents, whose relations at the micro-level generate emergent, collective behaviour, visible at the macro-level of inquiry. Many scientific disciplines, such as physics, biology and ecology have developed models to investigate how complex systems evolve (see, for instance, Boccara [33]). Within these, stochastic Cellular Automata models seem to be a useful tool in economic and social fields. The perceived complexity of organizations and markets, in which many agents interact with each other, has suggested the use of Cellular Automata also in innovation diffusion (see, for instance, Goldenberg and Efroni [34], Goldenberg *et al.* [32], Moldovan and Goldenberg [35], Goldenberg *et al.* [21]).

A Cellular Automaton consists of a finite number of individuals (or cells) that interact in a defined environment. Each cell can assume a particular state (for example, adopter, neutral) depending on its state in the previous time period and on the information received interacting with other cells. The evolution of a cell state is controlled by a predefined function called *transition rule*, which explicitly considers these interactions. The advantage provided by Cellular Automata models is the opportunity to observe the evolution of a given structure through the analysis of every single interaction between its components, representing another way, with respect to aggregate models, to deal with structural change and evolution. In this sense, Cellular Automata models may be powerful complements, rather than complete substitutes, of aggregate models for the analysis of life cycles and evolutionary patterns. In particular, the micro-level descriptive power of Cellular Automata could represent the conceptual introduction for new possible generalizations of the Bass model (see, for instance, Guseo and Guidolin [36]).

In this paper we use a cellular Automata Network (NA) for describing a network of interacting agents, who communicate between them information about a particular innovation. Then, we propose a two-stage modelling, representing the communication network and the proper adoption process that can occur only when there is sufficient knowledge about the innovation. In this case, the unit of analysis for Cellular Automata is represented by interpersonal links (edges): a *standard edge* between two different agents or a *reflexive edge* if we refer to a single agent taking into account “auto-communication”.

The state of an edge can be active or inactive: we suppose that the activation can occur either through institutional communication or through a word-of-mouth process that includes both interpersonal communication and signals due to previous adoptions.

We choose to analyze the state of edges rather than that of correspondent vertices, because we want to highlight that the construction of a collective knowledge relies on a process of social confirmation. Whatever the original source, institutional communication or word-of-mouth, the value and reliability of information is checked by individuals by making connections with others: we believe that this process of confirmation is a necessary phase for *information* to become *acquired knowledge*, that is awareness and persuasion to adopt.
Furthermore, we consider the possibility of edges' inactivation. This may happen either with a natural and autonomous decay process or with a negative word-of-mouth due to resistance to innovation effects. This represents a typical reaction to innovation for dissatisfaction or inadequate performance, whose effect may affect dynamically the market potential (see, for instance, Moldovan and Goldenberg [35]).

All these possibilities are described in a unique transition rule, able to represent the changing state of each edge. Once defined this communication network, that is the main driver of a dynamic market potential definition, the second stage of the model relates to the structure of the embedded purchase process, which is directly influenced by market potential evolution.

3. Evolution of Knowledge in a Communication Network

Let $G=(V,E)$ be a *finite* directed graph, where $V=\{1,2,...,i,...,N\}$ is a set of vertices whose cardinality is $N=c(V)$. The set E of ordered pairs (i,j), called *directed edges* or arcs, $E\subset V\times V$, depicts a subset of all the possible binary relationships between vertices V including reflexive relationships. From now on, we will use the simpler term *edge* to refer to a proper *directed edge*.
Due to possible limitations on *connectivity*, the cardinality of E is $U=c(E)\le N^2$. In social or physical systems these constraints may have natural interpretations based on large distances or accessibility restrictions that *a priori* exclude a possible link between two vertices. Each edge in E may assume, at time t, a state among a finite set of levels, $Q=\{0,1,2,...,K\}$. Here we assume a simple binary version, $Q=\{0,1\}$, i.e., an edge may be active, 1, when an information about an innovation is *transmitted* between vertices of an admissible edge or inactive, 0. We denote the *state*

of an edge (i,j) at time t with an indicator function $c(i,j;t)$. Function $c(i,j;t)$ equals 1 if and only if the edge (i,j) is active, otherwise is zero, in particular, if $(i,j) \notin E$.

Here we follow, only partially, some notations expressed for Automata Networks in Boccara *et al.* [27] and in Boccara and Fuks [37].

Let us define a rectangular centred neighbourhood $A_{(i,j)}$ around an edge (i,j) with *radii* ${}_1e_i$ and ${}_2e_j \in \hat{N}$ (the set of natural numbers including 0), i.e.,

$$A_{(i,j)} = \left\{ (r,s) \middle| i - {}_1e_i \le r \le i + {}_1e_i, j - {}_2e_j \le s \le j + {}_2e_j \right\}.$$

We assume that the transition rule $g(\cdot)$ governing network states is a function, possibly with stochastic components, of the edges' states of the neighbourhood $A_{(i,j)}, (i,j) \in E$ i.e., in expanded form,

$$\begin{aligned} c(i,j;t+1) = g(c(i - {}_1e_i, j - {}_2e_j;t), c(i - {}_1e_i + 1, j - {}_2e_j + 1;t), \ldots \\ \ldots, c(i + {}_1e_i, j + {}_2e - 1_j;t), c(i + {}_1e_i, j + {}_2e_j;t)) \end{aligned} \quad (3)$$

where $c(r,s;t) = 0$ if $(r,s) \notin E$. We assume here a discrete time $t \in \hat{N}$.

We may specify function $g(\cdot)$ as a combination of individual, local and external effects. A local effect on the state $c(i,j;t+1)$ of an edge (i,j) is determined by the joint influence of neighbouring edge states. More precisely, we define a *local pressure* (probability) of the system, $\sigma_c(i,j;t)$ on edge (i,j) to turn from an inactive status to an active one. This pressure depends on a flexible probability measure, $p_{n,m} \ge 0$, that allows a more general description of a neighbourhood, (i,j)-dependent.

$$\sigma_c(i,j;t) = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} c(i+n, j+m;t) p_{n,m}; \qquad \sum_{n,m} p_{n,m} = 1. \quad (4)$$

If we assume that this local pressure is *translational invariant*, we may consider the *mean field approximation* that excludes the local effect of distribution $p_{n,m}$,

$$\sigma_c(i,j;t) \cong \nu(t) = \sum_{i,j} \frac{c(i,j;t)}{U} \quad . \tag{5}$$

Let us define now a particular rule $g(\cdot)$ through a partially probabilistic specification, in order to describe the changing state of an edge as the result of individual characteristics, institutional information and local influences,

$$\begin{aligned} & c(i,j;t+1) = c(i,j;t) + Bi(1,p_c)I_{(c(i,j;t)=0)} \\ & +Bi(1,q_c\sigma_c(i,j;t))I_{(c(i,j;t)=0)} + \\ & -Bi(1,e_c)I_{(c(i,j;t)=1)} - Bi(1,w_c\sigma_c(i,j;t))I_{(c(i,j;t)=1)}. \end{aligned} \tag{6}$$

The second component of Equation (6), $Bi(1,p_c)I_{(c(i,j;t)=0)}$ depends on a binomial experiment, with parameter p_c, which is realizable only if the indicator function $I_{(c(i,j;t)=0)}$ is set to one, i.e., proposition $(c(i,j;t)=0)$ is true. The meaning of this component may be linked to the direct effect of external information like mass media communication channels and the change of state is possible, with probability p_c, only if this institutional communication reaches an edge (i,j). Note that institutional communication typically reaches some susceptible reflexive edge, (i,j).

The third component of Equation (6) considers the joint probability $q_c\sigma_c(i,j;t)$ that denotes the local pressure $\sigma_c(i,j;t)$ and the individual attitude for an imitative behaviour represented by parameter q_c. This second experiment is an opportunity strictly referred to standard edges (not reflexive) and expresses the idea that *imitative behaviour* is an *individual attitude* combined with a *local* geometry of evidence. The activation of these two components is strictly alternative, so that, if the first experiment changes the status of an edge, the second one is excluded and vice versa, the activation of an imitative behaviour forbids the innovative one.

The fourth component is a *decay effect* driven by a binomial $Bi(1,e_c)$ under the control of the correct state, $I_{(c(i,j;t)=1)}$, and describes the possible withdrawal from an active state representing a normal loss of information.

The fifth component represents a negative word-of-mouth driven by a binomial $Bi(1,w_c\sigma_c(i,j;t))$ under the control of the correct state, $I_{(c(i,j;t)=1)}$, and represents the possibility of inactivation due to opposite effects of local pressure producing resistance to innovation. Also these two exit rules are strictly alternative.

Once defined the stochastic transition rule (6) informing on how an edge may be activated, the second step is to recognize a convenient method to infer that emergent collective behaviour we aim to analyze.

In general, Cellular Automata are implemented through computer simulations generating a global behaviour from an individual (local) rule. The use of such techniques raises evident questions about the reliability of selected simulation parameters for which information is usually not available (see, for more details, [36]).

Alternatively we propose a local to global mapping considering a *mean field approximation* of transition rule (6). In this way we are able to infer a collective behaviour from observed aggregate data.

Let us consider, therefore, the average number of active edges within E at time t following the mean behaviour of transition rule (6),

$$U\nu(t+1) = U\left[\nu(t) + (p_c + q_c\nu(t))(1-\nu(t)) - e_c\nu(t) - w_c\nu^2(t)\right]. \tag{7}$$

We can approximate previous discrete time equation with a continuous Riccati equation, namely,

$$\nu'(t) = -(q_c + w_c)\nu^2(t) + (q_c - p_c - e_c)\nu(t) + p_c \tag{8}$$

and if we exclude e_c and w_c components, we obtain a standard Bass model. Solution $\nu(t)$ of Equation (8) is described in Appendix A.

Market potential definition

Function $U\nu(t)$, i.e., the number of informed edges, defines an aggregate temporal evolution of the knowledge about an innovation within the proposed communication network. This knowledge, based on active edges, is only a preliminary step in absorptive capacity definition using Cohen and Levinthal's (1990) terminology. We are interested in transforming this dynamic knowledge in a dynamic carrying capacity or market potential in order to define a potential boundary for the nested adoption process. This potential boundary is not a function of observed quantities, but is a latent structure that we cannot measure directly.

Without loss of generality we consider E as an approximate squared subset of $V \times V$, so that the positive squared root of $U\nu(t)$,

$$k(t) = \sqrt{U}\sqrt{\nu(t)} \tag{9}$$

depicts the number of *informed vertices*, that is the upper bound of the *market potential* $m(t)$ for the related process of innovation adoption by individuals describing the system, here represented as vertices of the graph $G = (V, E)$. Note that $k(t)$ is proportional to $\sqrt{\nu(t)}$ so that we may assume

$$m(t) = K\sqrt{\nu(t)} \tag{10}$$

as the actual market potential, where $K \leq \sqrt{U}$ if a vertex does not replicate a purchase. If replication is possible, K may be much greater than $\sqrt{U}$.

A simple generalization in $U\nu(t)$ transformation may be based on $\nu(t)^{\alpha}$ in order to take into account a possible dimensional collapse of $E \subset V \times V$. The proposed dynamic market potential in Equation (10) may be solved by recognizing in it a special version of Equation (19) (see Appendix A). For initial conditions $m(0) = 0$, $f(\cdot) = 1$ and $g(\cdot) = 1$ we obtain

$$m(t) = K\sqrt{\frac{1 - e^{-D_c t}}{\frac{1}{{}_c r_2} - \frac{1}{{}_c r_1} e^{-D_c t}}}, \qquad D_c = \sqrt{(q_c - p_c - e_c)^2 + 4(q_c + w_c)p_c} > 0, \tag{11}$$

where ${}_c r_i = (-(q_c - p_c - e_c) \pm D_c) / (-2(q_c + w_c))$, $i = 1,2$, with ${}_c r_2 > {}_c r_1$.

If, for instance, $e_c > 0$, then the limit of $m(t)$ for $t \to +\infty$ may be less than K.

INSERT FIGURE 1 ABOUT HERE

Vice versa, if communication effects are *persistent*, i.e. with no decay effect, $e_c = 0$, and no negative word-of-mouth, $w_c = 0$, then $D_c = q_c + p_c$ and ${}_c r_1 = -p_c / q_c$ $r_2 = 1$ so that

$$m(t) = K\sqrt{\frac{1-e^{-(p_c+q_c)t}}{1+\frac{q_c}{p_c}e^{-(p_c+q_c)t}}}\,. \qquad (12)$$

The limiting behaviour of $m(t)$ for $t \to +\infty$ equals the constant carrying capacity K.

In Figure (1) we represent different structures of a dynamic market potential under simplified Equation (12): for high values of parameters p_c and q_c we obtain asymptotically a constant potential K for $t > 0$ which is the typical market potential assumption in Bass models. On the contrary, low values of parameters p_c and q_c negatively affect $m(t)$, that reaches its asymptotic level much more slowly.

4. Co-evolution of the Diffusion of an Innovation

We denote the state of a vertex $i \in V$ at time t with indicator function $s(i;t)$. Following the same reasoning developed in Section 3, we define a transition rule for the description of an individual adoption process over time with the notation of a stochastic cellular automaton, i.e.,

$$\begin{aligned} s(i;t+1) &= s(i;t) + Bi(1,p_s)I_{(s(i;t)=0)} + Bi(1,q_s\sigma_s(i;t))I_{(s(i;t)=0)} + \\ &-Bi(1,r_s)I_{(s(i;t)=1)} + s(i;t)\frac{m'(t)}{m(t)} \end{aligned} \qquad (13)$$

The first four additive components of the right hand side member in Equation (13) may be interpreted following the same ideas of the previous section. In particular, the second component, $Bi(1,p_s)I_{(s(i;t)=0)}$ represents the innovative behaviour. Experiment $Bi(1,p_s)$ is performed with adoption innovative probability p_s if $s(i;t)=0$. The third component represents imitative contribution to adoption under a joint imitative probability based on two factors, an imitation coefficient, q_s, and a specific local pressure stimulating imitative adoption, $\sigma_s(i;t)$. The fourth component represents the possibility of disadoption with exit probability r_s. The fifth component, $s(i;t)\frac{m'(t)}{m(t)}$, describes an infinitesimal variation in the individual state due to the relative varying effect of $m(t)$ over time and is independent of K. For a constant market potential, $m(t)=M$ this component gives a null contribution.

This infinitesimal contribution depicts $s(i;t)\frac{m'(t)}{m(t)}$ as an interaction of the individual state with the global increasing or shrinking effect of the market potential. A possible extension may be based on a suitable weighting of the above interaction, i.e., $\alpha s(i;t)\frac{m'(t)}{m(t)}$. In the sequel we assume $\alpha = 1$.

The average behaviour of Equation (13) followed by a summation over all the states $s(i;t)$ within V is a discrete time aggregate *co-evolutionary model*

$$y(t+1) = y(t) + p_s(m(t) - y(t)) + q_s \frac{y(t)}{m(t)}(m(t) - y(t)) - r_s y(t) + y(t)\frac{m'(t)}{m(t)}. \tag{14}$$

A continuous approximation of Equation (14) is

$$y'(t) = m(t)\left\{-r_s \frac{y(t)}{m(t)} + \left(p_s + q_s \frac{y(t)}{m(t)}\right)\left(1 - \frac{y(t)}{m(t)}\right)\right\} + y(t)\frac{m'(t)}{m(t)}. \tag{15}$$

Evolution of an adoption process with marketing intervention strategies

An extension of Equation (15) is based on the modification of uniform dynamics due to exogenous interventions effects during the diffusion process. A similar approach is developed in Bass *et al.* [10] in the Generalized Bass Model (GBM).

We model this more flexible context through a multiplicative impact function, $x(t)$, that may incorporate exogenous factors, like marketing mix strategies, whose neutral level is $x(t) = 1 \forall t$, i.e.,

$$y'(t) = m(t)\left\{-r_s \frac{y(t)}{m(t)} + \left(p_s + q_s \frac{y(t)}{m(t)}\right)\left(1 - \frac{y(t)}{m(t)}\right)\right\} x(t) + y(t)\frac{m'(t)}{m(t)}. \tag{16}$$

Remind that $x(t)$ exerts its effect only on the first component of Equation (16), which is function of the residual market.

This nested two-phase model is a special Riccati equation analyzed in Appendix A. Note that in original GBM two special constraints apply: the decay component is excluded, $r_s y(t)/m(t) = 0$, and the market potential is constant, $m(t) = M$.

Solution of Equation (16) is presented in Section 5.

5. Statistical Co-evolutionary Modelling

The closed form solution of Equation (16), is determined on the basis of Equation (19) (see Appendix A), under an initial condition $C = 0$, for $g(\cdot) = m(\cdot)$ and $f(\cdot) = x(\cdot)$

$$y(t) = m(t)\frac{1 - e^{-D_s\int_0^\tau x(\tau)d\tau}}{\dfrac{1}{{}_s r_2} - \dfrac{1}{{}_s r_1} e^{-D_s\int_0^\tau x(\tau)d\tau}}, \qquad D_s = \sqrt{(q_s - p_s - r_s)^2 + 4 q_s p_s} > 0, \tag{17}$$

where ${}_s r_i = (-(q_s - p_s - e_s) \pm D_s)/(-2q_s)$, $i = 1,2$, with ${}_s r_2 > {}_s r_1$.

The time dependent market potential $m(t)$ penalizes with different emphases the evolution of the natural purchase process. In Figure 2 we represent two different communication frameworks. In case (a) we consider a good positive w-o-m, $q_c = 0.7$, and an absent effect of negative w-o-m, $w_c = 0$. In case (b) we have considered a negative w-o-m, 0.2, not compensated by a stronger positive component, $q_c = 0.9$. Case (b) exhibits a lower asymptotic market potential.

INSERT FIGURE 2 ABOUT HERE

The statistical implementation of model (17) may conceive different error structures. In a nonlinear regressive approach we consider a particular model for observations, $w(t) = y(t) + \varepsilon(t)$, with an i.i.d. residual $\varepsilon(t)$. A useful complementary approach is based on ARMAX representation with a standard nonlinear estimation as a first step (see, for instance, Guseo, [38], Guseo and Dalla Valle [39] and Guseo *et al.* [40]).

We shall notice that joint identifiability of parameters in Equation (8) is not possible because the autonomous Riccati Equation (19), under $f(\cdot) = g(\cdot) = 1$, is characterized by three independent parameters so that we have to evaluate which are the dominant effects or, more generally, we have to set one of the four parameters in Equation (8) to a specified level based on past experience. A reasonable choice may be e_c or w_c exclusion.

6. Pharmaceutical Drug Diffusion in Italian Geographic Areas

The problem of evaluating the role and the effectiveness of communication seems particularly impending in the case of new pharmaceutical drugs' diffusion, at least considering that recent years have seen dramatic increments in marketing spending by pharmaceutical companies.

The traditional communication strategy employed by pharmaceutical companies has generally focused on physicians through detailing, physician meetings and seminars, medical journals, advertising, samples and direct mail.

Among these promotional activities, detailing, that is sales representatives visiting physicians in order to provide information on new drugs, their usage, modes of therapy, prices, is the primary form of promotion.

There exists a broad literature on drugs' detailing effect on sales in medical science, marketing and economics. An interesting review of the research produced in this field is proposed in Manchanda and Honka [41], where the authors highlight the crucial effect of detailing in influencing the prescriptive behaviour of physicians. Interestingly, they document a wide consensus on the idea that when a new drug is launched, not much is known about its efficacy in practice, which makes detailing more effective: in particular, detailing plays an important role in the early and awareness-building phase of new drug's life cycle.

The literature on the effect of detailing over a new drug's sales has used both individual-physician level data to understand the prescription behaviour of physicians (see, for instance, Kamakura and Kossar [42]) and market-level data (Lilien et al. [43], Berndt et al. [44], Azoulay [45]).

Nevertheless, the acceptance of a new drug by a community relies on a wide knowledge-consensus about the benefits implied by its assumption. It has been demonstrated that various forces contribute to the construction of this knowledge stimulating w-o-m, like scientific research, medical community, institutions, profit and non-profit organizations, consumers' associations, magazines, journals and the Internet.

The diffusion of a new drug is a social phenomenon that ultimately depends on the decisions of end-users supported by various aspects of w-o-m. In this sense it is reasonable to use sales data: what matters is the final purchase, which may be just partially mediated by physicians' prescriptions.

We have tested the performance of our model with the weekly data of the diffusion of five new pharmaceutical drugs in Italy. Our aim is to measure the extent of different communication channels on new drugs' sales in different markets.
The data, provided by IMS Health, cover the period between August 2005 and July 2007 with a spatial disaggregation by areas (“NordEst”, “NordOvest”, “Centro”, “Sud”). This type of information allows interesting comparisons between areas and considerations on their different level of receptiveness.

INSERT FIGURE 3 ABOUT HERE

For simplicity and space reasons we examine only “NordEst” and “Centro” and consider only the case of a new drug, here denoted by “FOL”. The results and implications of this specific case apply to the other four drugs we have analyzed. “FOL” has been introduced in August 2005 and prescribed by physicians to prevent fetus malformations. The assumption of “FOL” by expectant mothers has been also recommended by the Italian Department of Health.
In April 2005 the Italian Agency for Pharmaceuticals (AIFA) approved the introduction of 94 new drugs, whose cost is entirely covered by the National Health Service: Within these, we find “FOL”, which is based on Folic Acid. This has represented an institutional acknowledgement of the central role of Folic Acid in preventing birth defects and an important support for all public actions aimed at promoting its assumption by expectant mothers. Among these actions, a very relevant one is the foundation of the Italian Network for the Promotion of Folic Acid.

Observing Figure (3) we may notice that the instantaneous data on the diffusion of this drug by areas show a good trend for “Centro” of Italy, while the diffusion process appears less developed in “NordEst”. We therefore choose to compare the case of “Centro” with that of “NordEst” and we apply our model to the corresponding cumulative data to check how these two areas have developed their respective market potentials, as result of effective communication strategies. In both cases we apply the model in its reduced form, without considering exit rates (parameters $w_c = e_c = r_s = 0$) and exogenous interventions (function $x(t) = 1$),

$$w(t) = K \sqrt{\frac{1 - e^{-(p_c + q_c)t}}{1 + \frac{q_c}{p_c} e^{-(p_c + q_c)t}}} \frac{1 - e^{-(p_s + q_s)t}}{1 + \frac{q}{p} e^{-(p_s + q_s)t}} + \varepsilon(t) \tag{18}$$

In Table 1 we summarize the estimation results for both geographical areas under a standard nonlinear least squares approach (Levemberg-Marquardt; see, for instance, Seber and Wild [46]) in our co-evolutionary model as expressed in Equation (18).

INSERT TABLE 1 ABOUT HERE

As we may notice by the values of the determination index, R_1^2, the model presents very high levels of global fitting in both cases and all the involved parameters are significant. Remind that in S-shaped models it is quite common to obtain high levels for the determination index R^2. For instance, $R^2 = 0.95$ is a low goodness-of-fit value because the competing model is too elementary: the constant one.

Interestingly, we observe that the communication parameters have a higher value in the area of "NordEst", while they are much lower in "Centro", despite the apparently better diffusion process in this area.

How do these two communication structures affect their respective diffusion processes? We may appreciate this aspect in graphical terms.

INSERT FIGURE 4 ABOUT HERE

Figure (4) shows that the market potential in "NordEst" has reached its saturation level faster than that of "Centro", indicating a greater effectiveness of communication strategies. Indeed, this has a visible effect on the diffusion process, which is evidently faster in "NordEst". Notice that the speed of the diffusion process results from the combined effect of communication and adoption parameters, so that the different performance of "NordEst" and "Centro" is also due to different adoption parameters. However, we can easily check from the above estimation results that just the imitative adoption pattern, represented by parameter q_s, is significantly different between the two areas, while the innovative one, p_s, is essentially equal. Besides, we may observe the faster take-off in "NordEst" as a direct effect of a faster generation of the market potential.

The results offered by this application make conclude that the market for this drug in "Centro" is less developed than should/could do and clearly suggest a change in management choices, able to improve the effectiveness of communication efforts in this area.

Are the differences between the two areas really significant or not? We may simply test this aspect observing the confidence intervals of the estimated parameters: for example, if we consider parameter estimates for “NordEst” we will see that, except for parameter p_s, the others have a higher value than that of the upper confidence interval of correspondent parameters for “Centro”. This confirms that the difference is significant and may be used for managerial purposes.

Our model also provides evident improvements in forecasting terms, as one may see comparing its performance with that of a standard Bass model, BM, which is a nested special case.
With this reduced modelling applied to the above cases, differences in estimation may be easily appreciated. In Table (2) we summarize the estimation results for both geographical areas under the standard nonlinear least squares approach.

INSERT TABLE 2 ABOUT HERE

At a glance we see that the application of a standard Bass model offers a weaker performance, as shown by the values of the determination indexes, ${R_2}^2$, that in these cases are much lower as we will clarify in a formal way below.
However, the most surprising result are the estimated values (underestimates) of the market potential m or K, that are significantly different from those obtained using our model.
Note, in particular for “Centro”, that the confidence interval of the Bass model in K is 358304 to 399833 while the confidence interval in K for our co-evolutionary model is 638583 to 763867.

We argue that this is due to the well-known problem of the Bass model, which tends to overestimate data in the first part of the life cycle -exactly not recognizing that *incubation period* typical of early stages of diffusion- and counterbalances this lack-of-fit with an underestimation in the final part of the diffusion process.
As a consequence, not taking into account the incubation period and its effect on the market potential, may seriously affect forecasting in terms of market potentials.

In order to test the global significance of our extended models, as compared with the BM ones, we can compute a squared multiple partial correlation coefficient, $P^2 = ({R_1}^2 - {R_2}^2)/(1 - {R_2}^2)$, which is normalized within the interval [0,1], and the corresponding F-ratio, i.e. $F = P^2(N-k)/\left[(1-P^2)s\right]$ where N is the number of observations, k the parameter cardinality of the extended model and, $k-s$ is the number of parameters of a nested model (a common critical value for F-ratio is 4).

A direct comparison between R_1^2 and R_2^2 based on a simple difference $R_1^2 - R_2^2$, is misleading because not normalized.

For “NordEst” area we obtain $P^2{}_{NE} = 0.9352$ and $F_{NE} \cong 671$. Similar results for “Centro” area, i.e., $P^2{}_C = 0.9708$ and $F_C \cong 1546$. Both co-evolutionary extensions are strongly significant if compared with a baseline standard Bass model. We can appreciate the efficient performance of the co-evolutionary model with respect to the standard Bass one by examining, for instance, the corresponding graphical aspects of competing models for "Centro" (see Figure 5).

INSERT FIGURE 5 ABOUT HERE

We observe, in particular, the presence of a “local depression” in data which strongly departs from a classical bell-shaped diffusion (BM or, more generally, a NUI model, see Easingwood *et al.* [47]): this effect is quite perfectly absorbed by our model and has been observed in parallel cases.

We have highlighted quite outstanding determination indexes in Table 1. Nevertheless, the Durbin-Watson statistic in both cases (0.476 and 0.556) suggests the presence of autocorrelated residuals that may be seen observing the original instantaneous data plotted in Figure 3.
This problem may be overcome by implementing an appropriate ARMAX procedure. The main results are outlined in Table 3 for the model PREfolcoevC (“Centro” area) and in Table 4 for the model PREfolcoevNE (“NordEst” area).

INSERT TABLE 3 ABOUT HERE

The significance of ARMAX sharpening with respect to the co-evolutionary model PREfolcoevC of “Centro” area may be easily determined via a squared multiple partial correlation coefficient, $P^2 = (R_3^2 - R_1^2)/(1 - R_1^2)$ and the corresponding *F*-ratio. In particular, we obtain $P_C^2 = 0.62166$ and $F_C \cong 37$.
Similarly, for “NordEst” area we obtain $P_{NE}^2 = 0.55188$ and $F_{NE} \cong 37$.

INSERT FIGURE 6 ABOUT HERE

Both ARMAX extensions are significant. For a graphical comparison among actual data, co-evolutionary modelling and ARMAX sharpening for “FOL” weekly sold (non-cumulative) packages in “NordEst” and “Centro”, see Figures 6 and 7.

INSERT TABLE 4 ABOUT HERE

INSERT FIGURE 7 ABOUT HERE

7. Final Remarks and Discussion

This paper faces different aspects in innovation diffusion modelling by combining theoretical, technical and applied aspects on communication dynamics and adoption processes.

Here we summarize some of these:

a. Innovation diffusion is not a univariate adoption process over time. We present a binary model for an adoption process nested in a communication network that evolves dynamically and generates the corresponding non-constant market potential;

b. We build a communication network as a necessary phase in determining the evolution of a *prior related knowledge*, which is, using Cohen and Levinthal's (1990) terminology, the basic element for developing an *absorptive capacity*;

c. Our two-phase model is a particular technical specification of the above general ideas: Cellular Automata and Network Automata are a simple and effective tool for representing both the communication network evolution and the nested adoption process;

d. In our model we assume that the communication network is not observable. In general we do not have precise information about how agents communicate between them and the network we consider has a virtual structure. However, we are not interested in determining detailed particular shapes of the actual network. Our focus is on an aggregate transformation of this network, i.e., the concrete market potential $m(t)$;

e. To represent the communication network we have focused on the sources of information through which an edge is activated. We shall notice that in transition rule (6) we allow the activation of an edge if both vertices receive the information from an external source or from an internal one (neighbouring pressure). In other words, we do not consider the possibility of a mixed connection, when a vertex is informed by an external source and the other by an internal one. Although the mixed case is possible, we prefer to focus on *pure* *cases*, observing that in general a person tends to get confirmation of his believes by someone with similar references. We have therefore chosen to characterize our network as a set of interpersonal connections based on different informational habits. Incidentally, we observe that this network does not consider the spatial dimension of connections and does not make assumptions on the strength of relationships: in this sense, our approach differs from that inspired by Granovetter [48] and his theory of strong and weak ties, where network dynamics are described as a function of the strength of interpersonal ties. In our model we only require that the activation of an edge may happen if two vertices share the same informational reference (institutional communication or word-of-mouth). In this way we are able to maintain a good level of generality, as the application of our model just needs aggregate sales data with no explicit information about spatial connectivity among agents;

f. With a *mean field approximation* we have reformulated a Complex Systems representation in a dual tractable differential one, see Equations (8), (9) and (16);

g. In our model, especially with reference to the Riccati Equation (16), functions $m(t)$ and $x(t)$ are independent tools. The effect of intervention function $x(t)$ modifies the *temporal path* of diffusion by locally expanding or shrinking purchases within a "balance equation constraint". Instead, the market potential, $m(t)$, controls and modifies the *size* (scale) of this process, expressed in terms of the absolute amount of sales. This specification is useful to avoid theoretical misunderstandings between these different and separable effects;

h. Modelling the market potential as a function of a communication network implies that marketing efforts should be concentrated during very early stages of diffusion. Effective communication strategies allow a fast development of the market potential, reducing the negative impact of the incubation phase for those goods that do not present strong network externalities;

i. As we have seen comparing the performance of our model with that of a BM, not recognizing the effect of incubation may affect forecasting especially in terms of market potential underestimation. Since our model yields higher values of R^2 and *F*-tests confirm the significance of this extension, we have proposed it as a useful and usable tool for market diagnostics and forecasting for new drugs. We suppose that this model may be applied to other stand alone goods' diffusions;

j. The possibility to isolate communication dynamics within a diffusion process using aggregate *sales data*, suggests interesting comparisons not only between different markets with respect to the same good, but also within the same market between competing goods by examining the expected results of marketing efforts and investments with the real market response to these expressed in terms of actual sales;

k. The proposed application gives some insights on the role of statistics in analyzing evolving time series within a life cycle context. In particular, we observe that in this specific application the *mean field approximation*, that allows an interesting aggregate description of a Complex Systems representation, does not consider effects of a supposed (not observed) heterogeneity of individuals (or their purchases). Nevertheless, an ARMAX sharpening, applied as a second step after a nonlinear least squares procedure, completes inference in a satisfactory way.

Appendix A

Riccati Equation, a Special Case

Let us consider the following special Riccati equation in (X,Y) real space

$$y'_x = a\frac{f(x)}{g(x)}y^2 + \left(bf(x) + \frac{g'(x)}{g(x)}\right)y + cf(x)g(x) \tag{19}$$

where $a,b,c \in R, D = \sqrt{b^2 - 4ac} > 0$ and $g(x) \neq 0$, $f(x)$ are real functions.

We note that this special version of non-autonomous Riccati equation is not examined in the well-known *Handbook* by [49].

The analysis proposed in the sequel represents a contribution to Polyanin's Cataloge.

An equivalent form of Equation (19) is

$$\frac{y'_x g(x) - g'(x)y}{g(x)} = \left(\frac{a}{g(x)}y^2 + by + cg(x)\right) f(x), \tag{20}$$

or

$$\frac{y'_x g(x) - g'(x)y}{g^2(x)} = \left(a\left(\frac{y}{g(x)}\right)^2 + b\left(\frac{y}{g(x)}\right) + c\right) f(x). \tag{21}$$

With a simple substitution, i.e. $z = y / g(x)$, we have

$$z' = (az^2 + bz + c)f(x) \tag{22}$$

for which a general solution is attainable.

Let us consider the real roots of equation $az^2 + bz + c = 0$, i.e. $r_i = (-b \pm D)/2a \in R, i = 1,2$ where $D = a(r_2 - r_1) = \sqrt{b^2 - 4ac} > 0$ so that equation (22) may be represented as follows

$$z' = a(z - r_1)(z - r_2)f(x). \tag{23}$$

Consider the substitution $\dot{z} = z - r_2$ with $\dot{z}' = z'$ and initial conditions $z(0) = C$ or $\dot{z}(0) = C - r_2$. Dividing both members of transformed previous equation by $\dot{z}^2$, we attain $\frac{\dot{z}'}{\dot{z}^2} = a(\dot{z} + r_2 - r_1)\frac{1}{\dot{z}} f(x)$, or $\frac{\dot{z}'}{\dot{z}^2} = \left\{a(r_2 - r_1)\frac{1}{\dot{z}} + a\right\} f(x)$.

A further substitution, $\hat{z} = \frac{1}{\dot{z}}$, with $\hat{z}' = -\frac{\dot{z}'}{\dot{z}^2}$ and initial condition $\hat{z}(0) = \frac{1}{C - r_2}$, yields equation

$$-\hat{z}' = \left\{a(r_2 - r_1)\hat{z} + a\right\} f(x), \tag{24}$$

which may be integrated as a linear first order equation (see, e.g. Apostol [50], p.31).

Its solution is

$$\hat{z} = \frac{1}{C - r_2} G(x) + G(x) a \int_0^x f(\tau) e^{-a(r_2 - r_1)\int_0^\tau f(\xi) d\xi} d\tau, \tag{25}$$

where $G(x) = e^{a(r_2 - r_1)\int_0^x f(\tau) d\tau}$ or equivalently $G(x) = e^{D\int_0^x f(\tau) d\tau}$ so that

$$\begin{aligned} \hat{z} &= \frac{1}{C - r_2} G(x) + G(x) a \left[-\frac{1}{D} e^{-D\int_0^x f(\xi) d\xi} + \frac{1}{D} \right] \\ &= \frac{G(x)}{C - r_2} - \frac{1}{(r_2 - r_1)}[1 - G(x)] = \frac{r_2 - r_1 G(x) - C(1 - G(x))}{(C - r_2)(r_2 - r_1)}. \end{aligned} \tag{26}$$

Let us express solution (26) in terms of the initial variable, $z = \frac{1}{\hat{z}} + r_2$,

$$\begin{aligned} z &= r_2 + \frac{(C - r_2)(r_2 - r_1)}{r_2 - r_1 G(x) - C(1 - G(x))} \\ &= \frac{r_1 r_2 (1 - G(x)) - C(r_1 - r_2 G(x))}{r_2 - r_1 G(x) - C(1 - G(x))}. \end{aligned} \tag{27}$$

We obtain the general solution of Equation (19) in a direct way, i.e.

$$y(x) = g(x) \frac{r_1 r_2 (1 - G(x)) - C(r_1 - r_2 G(x))}{r_2 - r_1 G(x) - C(1 - G(x))}. \tag{28}$$

If the initial condition is set to zero, $C = 0$, we obtain

$$y(x) = g(x)\frac{1-G^{-1}(x)}{\frac{1}{r_2}-\frac{1}{r_1}G^{-1}(x)} = g(x)\frac{1-e^{-D\int_0^x f(\tau)d\tau}}{\frac{1}{r_2}-\frac{1}{r_1}e^{-D\int_0^x f(\tau)d\tau}}. \tag{29}$$

If $\lim_{x\to\infty}\int_0^x f(\tau)d\tau = +\infty$, we attain an interesting limiting behaviour of $y(x)$, i.e., $\lim_{x\to\infty} y(x) = r_2 \lim_{x\to\infty} g(x)$.

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References

[1] H.A. Gatignon, T.S. Robertson, A Propositional Inventory for New Diffusion Research, Journal of Consumer Research 11, (1985) 849-867.

[2] V. Mahajan, E. Muller, Innovation Diffusion and New Product Growth Models in Marketing, Journal of Marketing 43, (1979) 55-68.

[3] V. Mahajan, E. Muller, F.M. Bass, New Product Diffusion Models in Marketing: A Review and Directions for Future Research, Journal of Marketing 54, (1990) 1-26.

[4] V. Mahajan, E. Muller, Y. Wind, New-Product Diffusion Models, Springer Science + Business Media, New York, 2000.

[5] N. Meade, T. Islam, Modelling and forecasting the diffusion of innovation - A 25-year review, International Journal of Forecasting 22, (2006) 519-545.

[6] E. Muller, R. Peres, V. Mahajan, Innovation Diffusion and New Product Growth: Beyond a Theory of Communications, working paper, 2007, (http://www.hitechmarkets.net/files/ReviewPaperApril2007final.pdf)

[7] L.A. Fourt, J.W. Woodlock, Early Prediction of Market Success for New Grocery Products, Journal of Marketing 25, (1960) 31-38.

[8] E. Mansfield, Technical Change and the Rate of Imitation, Econometrica 29(4), (1961) 741-766.

[9] F.M. Bass, A new product growth model for consumer durables, Management Science 15, (1969) 215-227.

[10] F.M. Bass, T. Krishnan, D. Jain, Why the Bass model fits without decision variables, Marketing Science 13, (1994) 203-223.

[11] V. Mahajan, R.A. Peterson, Innovation Diffusion in a Dynamic Potential Adopter Population, Management Science 24(15), (1978) 1589-1597.

[12] D. Horsky, A diffusion model incorporating product benefits, price, income and information, Marketing Science 9, (1990) 342-365.

[13] N. Kamakura, S. Balasubramanian, Long-Term View of the Diffusion of Durables: A Study of the Role of Price and Adoption Influence Processes via Tests of Nested Models, International Journal of Research in Marketing 5, (1988) 1-13.

[14] H.I. Mesak, A.F. Darat, Optimal pricing of new subscriber services under interdependent adoption processes, Journal of Service Research 5(3), (2002) 140-153

[15] M. Sharif, K. Ramanathan, Binomial Innovation Diffusion Models with Dynamic Potential Adopter Population, Technological Forecasting and Social Change 20, (1981) 63-87.

[16] D.C. Jain, R.C. Rao, Effect of Price on the Demand of Durables: Modeling, Estimation and Findings, Journal of Business and Economic Statistics 8, 1990 163-170.

[17] P.M. Parker, Price Elasticity Dynamics Over the Adoption Life Cycle, Journal of Marketing Research XXIX, (1992) 358-367.

[18] P.M. Parker, Choosing among diffusion models: some empirical evidence, Marketing Letters 4(1), (1993) 81-94.

[19] S.K. Rao, An empirical comparison of sales forecasting models, Journal of Product Innovation Management 2, (1985) 232-242.

[20] N. Kim, E. Bridges, R.K. Srivastava, A simultaneous model for innovative product category sales diffusion and competitive dynamics, International Journal of Research in Marketing 16(2), (1999) 95-111.

[21] J. Goldenberg, L. Barak, E. Muller, The Chilling Effect of Network Externalities on New Product Growth, (2005), (working paper). Tel Aviv University; www.complexmarkets.com

[22] F. Centrone, A. Goia, E. Salinelli, Demographic Processes in a Model of Innovation Diffusion with Dynamic Market, Technological Forecasting and Social Change 74(3), (2007) 247-266.

[23] S. Kalish, A New Product Adoption Model with Pricing, Advertising and Uncertainty, Management Science 31, (1985) 1569-1585.

[24] P. Meyer, J.H. Ausubel, Carrying capacity: a model with logistically varying limits, Technological Forecasting and Social Change 61(3), (1999), 209-214.

[25] M.S. Sawhney, J. Eliashberg, A Parsimonious Model for Forecasting Gross Box-Office Revenues of Motion Pictures, Marketing Science 15(2), (1996) 113-131.

[26] W.M. Cohen, D.A. Levinthal, Absorptive Capacity: A new Perspective on Learning and Innovation, Administrative Science Quarterly 35, (1990) 128-152.

[27] N. Boccara, H. Fuks, S. Geurten, A New Class of Automata Networks, Physica D 103, (1997) 145-154.

[28] V. Mahajan, E. Muller, R.A. Kerin. Introduction Strategy for New Products with Positive and Negative Word-of-Mouth, Management Science 30, (1984) 1389-1404.

[29] J. Eliashberg, J. Jonker, M. Sawhney, B. Wierenga, MOVIEMOD: An Implementable Decision-Support System for Prerelease Market Evaluation of Motion Pictures, Marketing Science 19(3), (2000) 226-243.

[30] R. Chatterjee, J. Eliashberg, The Innovation Diffusion Process in a Heterogeneous Population: A Micromodeling Approach, Management Science 36, (1990) 1057-79.

[31] J.H. Roberts, J.M. Lattin, Disaggregate-Level Diffusion Models, in V. Mahajan, E. Muller, Y. Wind, (Eds.) New-Product Diffusion Models, Springer Science + Business Media, New York, (2000) 207-236.

[32] J. Goldenberg, B. Libai, E. Muller, Talk of the Network: A Complex Systems Look at the Underlying Process of Word-of-Mouth, Marketing Letters 12(3), (2001) 211-223.

[33] N. Boccara, Modeling Complex Systems, Springer-Verlag, New York, 2004.

[34] J. Goldenberg, S. Efroni, Using cellular automata modeling of emergence of innovations, Technological Forecasting and Social Change 68(3), (2001) 293-308.

[35] S. Moldovan, J. Goldenberg, Cellular automata modeling of resistance to innovations: Effects and solutions, Technological Forecasting and Social Change 71, (2004) 425-442.

[36] R. Guseo, M. Guidolin, Cellular Automata and Riccati Equation Models for Diffusion of Innovations, Statistical Methods and Applications, 17(3) (2008), 291-308, http://dx.doi.org/10.1007/s10260-007-0059-3

[37] N. Boccara, H. Fuks, Modeling diffusion of innovations with probabilistic cellular automata. M. Delorme, J. Mazoyer, eds. *Cellular Automata: A Parallel Model*, Kluwer, Dordrecht 1999. ISBN 0-7023-5493-1; (online) arXiv: adap-org/9705004.

[38] R. Guseo, Interventi strategici e aspetti competitivi nel ciclo di vita di innovazioni, Working Paper Series, 11, Department of Statistical Sciences, University of Padova, 2004 b.

[39] R. Guseo, A. Dalla Valle, Oil and Gas Depletion: Diffusion Models and Forecasting under Strategic Intervention, Statistical Methods and Applications 14, (2005) 375-387.

[40] R. Guseo, A. Dalla Valle, M. Guidolin, World Oil Depletion Models: Price Effects Compared with Strategic or Technological Interventions, Technological Forecasting and Social Change 74(4), (2007) 452-469.

[41] P. Manchanda, E. Honka, The effects and role of direct-to-physician marketing in the pharmaceutical industry: an integrative review, Yale Journal of Health Policy, Law and Ethics 5(2), (2005) 785-822.

[42] W. Kamakura, B. Kossar, A factor-analytic split hazard model for database marketing, W.P. Fuqua School of Business, (1998) Duke University.

[43] G.L. Lilien, A.G. Rao, S. Kalish, Bayesian estimation and control of detailing effort in a repeat purchase environment, Management Science 27, (1981) 493-507.

[44] E.R. Berndt, R.S. Pindick, P. Azoulay, Consumption Externalities and Diffusion in Pharmaceutical Markets: Antiulcer Drugs, The Journal of Industrial Economics 51(2), (2003) 243-270.

[45] P. Azoulay, Do Pharmaceutical Sales Respond to Scientific Evidence? Journal of Economics & Management Strategy 11(4), (2002) 551-594.

[46] G. Seber, C. Wild, Nonlinear Regression, Wiley, New York, 1989.

[47] C.J Easingwood, V. Mahajan, E. Muller, A nonuniform Influence Innovation Diffusion Model of New Product Acceptance, Marketing Science 2(3), (1983) 273-295.

[48] M.S. Granovetter, The Strength of Weak Ties, The American Journal of Sociology 78(6), (1973) 1360-1380.

[49] A.D. Polyanin, V.F. Zaitsev, Handbook of Exact Solutions for Ordinary Differential Equations, 2nd Edition, Chapman & Hall/CRC, Boca Raton, 2003. http://eqworld.ipmnet.ru/en/solutions/ode/ode0123.pdf

[50] T.M. Apostol, Calcolo: Analisi 2, Vol. 3. Bollati Boringhieri, Torino, 1978.

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Table 1. Pharmaceutical drug's diffusion. Parameters' estimates of co-evolutionary models for "FOL" in "Centro" and "NordEst" areas of Italy with no exit rule. ( ) marginal linearized asymptotic confidence limits.

"Centro"						
K	q_c	p_c	q_s	p_s	R_1^2	$D-W$
763867 (638683) (889051)	0.08190 (0.07342) (0.09038)	0.01192 (0.01099) (0.01229)	0.01728 (0.01549) (0.01908)	0.00175 (0.00154) (0.00197)	0.999967 SSE: [1.99934E7]	0.476
"NordEst"						
K	q_c	p_c	q_s	p_s	R_1^2	$D-W$
339352 (320070) (358633)	0.09430 (0.07664) (0.11196)	0.01969 (0.01657) (0.02282)	0.02487 (0.02385) (0.02590)	0.00175 (0.00170) (0.00180)	0.999961 SSE: [8.39324E6]	0.556

Table 2. Pharmaceutical drug's diffusion. Parameters' estimates of standard Bass models for "FOL" in "Centro" and "NordEst" areas of Italy. ( ) marginal linearized asymptotic 95% confidence limits.

"Centro"				
K	q	p	R_2^2	$D-W$
379069 (358304) (399833)	0.03222 (0.03037) (0.03406)	0.00237 (0.00229) (0.00244)	0.99887 SSE: [6.77495E8]	0.025
"NordEst"				
K	q	p	R_2^2	$D-W$
256519 (243093) (269946)	0.03142 (0.03007) (0.03276)	0.001923 (0.00187) (0.00198)	0.999398 SSE: [1.28046E8]	0.050

Table 3. FOL-Centro, Italy: Co-evolutionary cumulative model with no exit rule and ARMAX (2,0,1) sharpening. ( ) *t*-statistic; [ ] p-values.

AR(1)	AR(2)	MA(1)	PREfolcoevC	mean	SSE
1.68001 (45.3592) [0.000000]	-0.821748 (-37.986) [0.000000]	0.884392 (8.30299) [0.000000]	0.14191 (8.30299) [0.000000]	783.875 (4.33934) [0.000036]	7.564208E6 d.f. 94 R^2_3 = 0.999987

Table 4. FOL-Nordest, Italy: Co-evolutionary cumulative model with no exit rule and ARMAX (2,0,0) sharpening. ( ) *t*-statistic; [ ] p-values.

AR(1)	AR(2)	PREfolcoevC	mean	SSE
1.00659 (10.6144) [0.000000]	-0.489008 (-7.48746) [0.000000]	0.482926 (7.94794) [0.000000]	38.4709 (0.243694) [0.807993]	3.761126E6 d.f. 95 R_3^2 = 0.999982

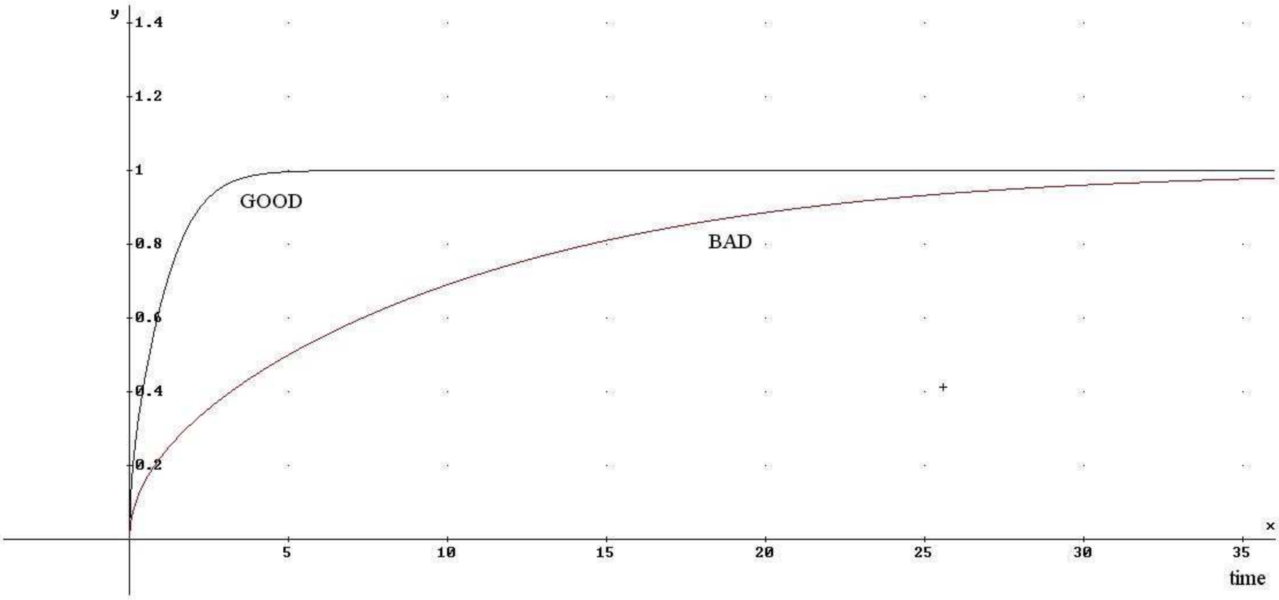

Fig. 1. Two normalized ($K=1$) dynamic market potentials over time (x). Good communication: $p_c = 0.15$, $q_c = 0.90$; Bad communication: $p_c = 0.01$, $q_c = 0.06$.

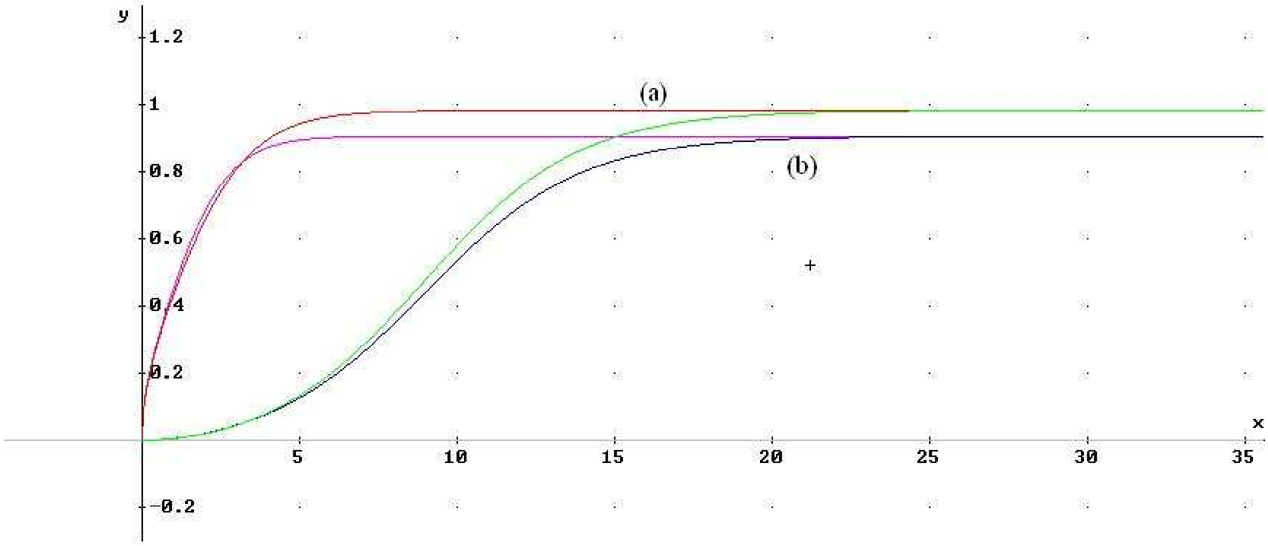

Fig.2. Two different dynamic market potentials and corresponding adoption processes over time (x). Common adoption parameters: $q_s = 0.4, p_s = 0.01, r_s = 0$; Common communication parameters: $K = 1$, $p_c = 0.15, e_c = 0.03$. Special cases: Case (a): $q_c = 0.7, w_c = 0$, Case (b): $q_c = 0.9, w_c = 0.2$.

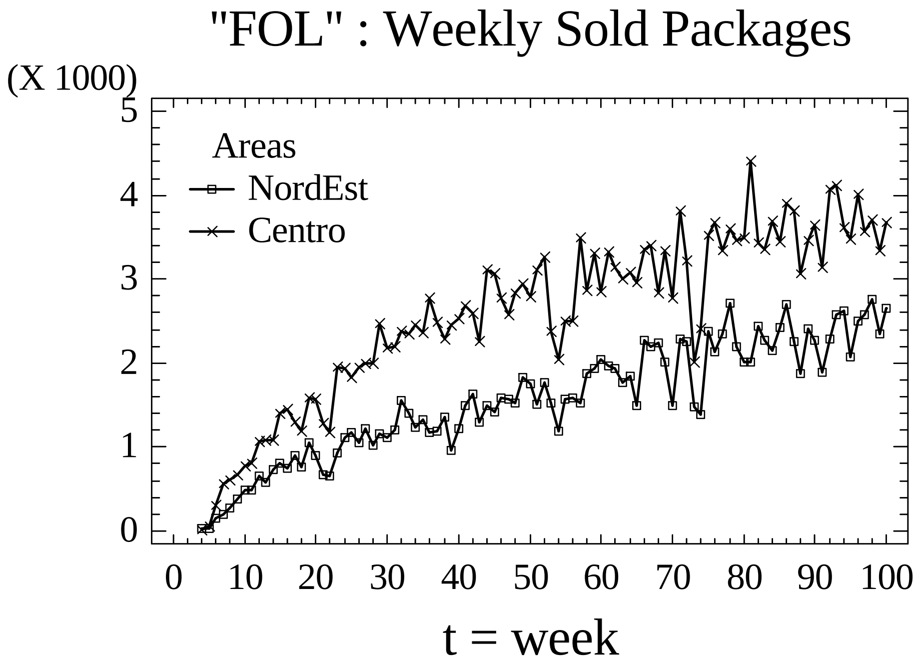

Fig.3. “FOL”: number of weekly sold packages in two geographic areas of Italy.

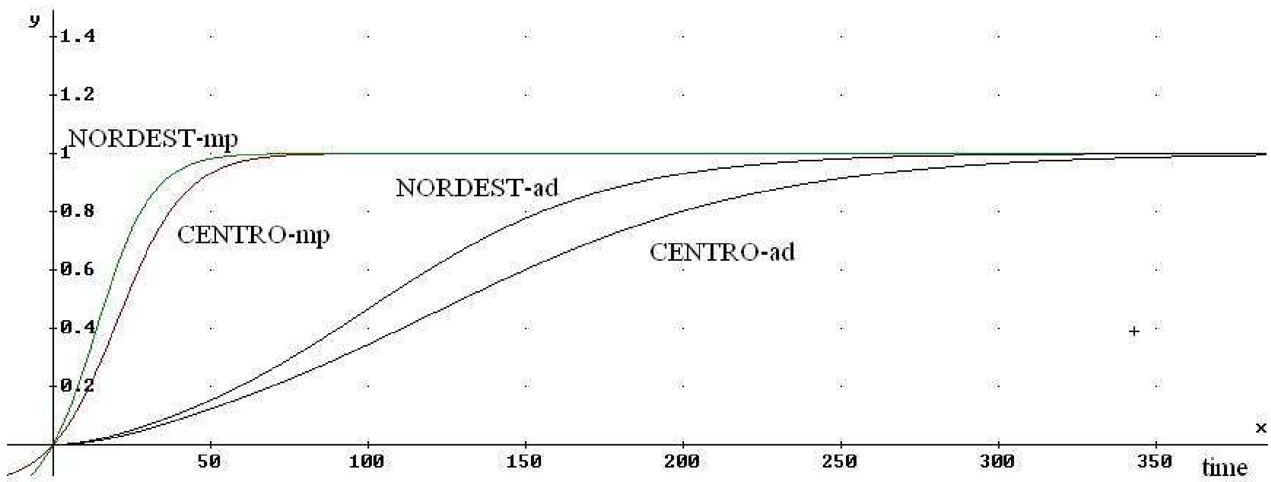

Fig. 4. “FOL”: Normalized comparison $(K = 1)$ between market potential and corresponding adoption process over time, *x*, (week) in two Italian areas: “NordEst” and “Centro”.

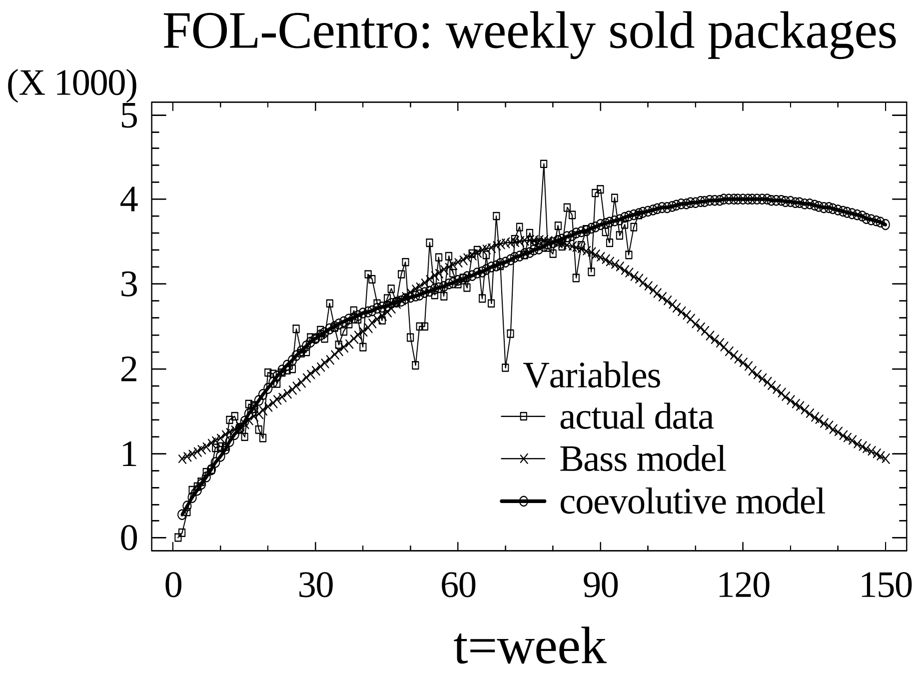

Fig.5. FOL-Centro, Italy: Co-evolutionary non-cumulative model with no exit rule, standard Bass model and weekly “FOL” non cumulative sales’ data.

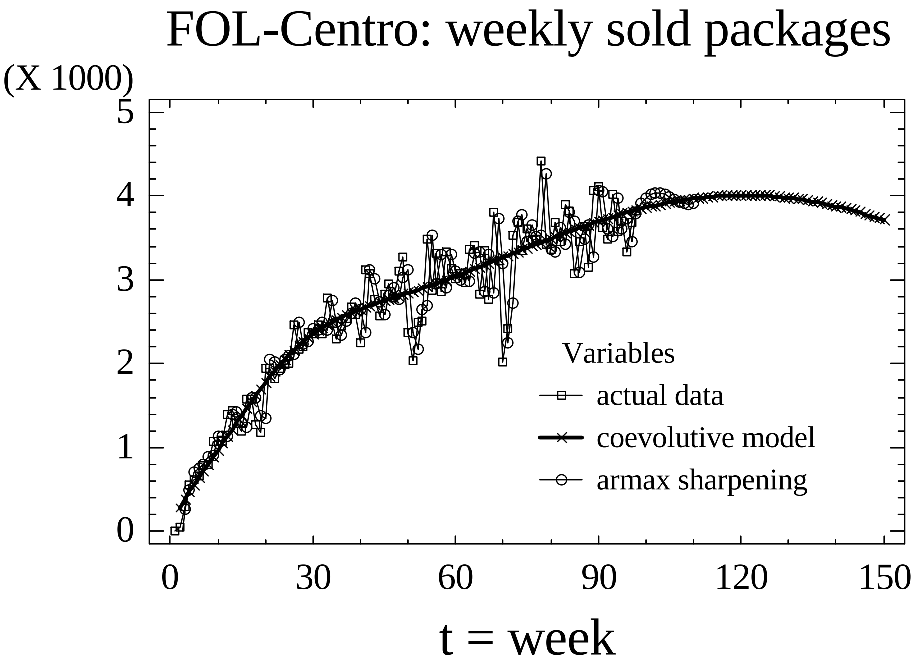

Figure 6. FOL-Centro, Italy: Co-evolutionary non-cumulative model with no exit rule, ARMAX sharpening and weekly "FOL" non-cumulative sales' data.

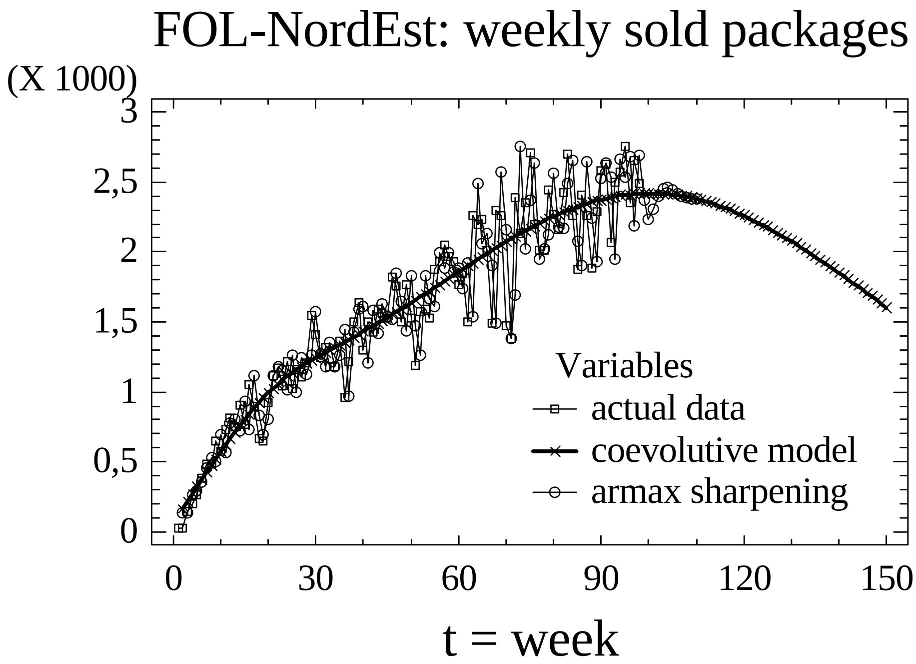

Fig. 7. FOL-NordEst, Italy: Co-evolutionary non-cumulative model with no exit rule, ARMAX sharpening and weekly "FOL" non-cumulative sales' data.