Competition Modelling in Multi-Innovation Diffusions: Multivariate Cellular Automata and Differential Approaches

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Abstract

La diffusione di innovazioni in un sistema sociale è stata affrontata da vari punti di vista negli ultimi trent'anni: matematici, fisici, esperti di marketing quantitativo, statistici, ingegneri e analisti di sistemi, biologi, economisti evolutivi, epidemiologi ed esperti di sistemi ecologici hanno dato sicuri contributi. Qui si propone una serie di ampliamenti di un modello di Guseo e Bonaldo (1991), lavoro che per la prima volta generalizza il modello univariato di Bass (1969) nella direzione bivariata. La trattazione utilizza un automa cellulare bivariato a tempo discreto e la sua successiva approssimazione continua. I risultati principali si riferiscono alle soluzioni in forma chiusa e alle corrispondenti estensioni per tener conto di una competizione sincronica e diacronica con eventuali interventi.

1. Introduction

In Guseo and Guidolin (2007a) it has been proved, for a univariate case and under a "mean field approximation", that there exists a differential dual representation of a particular Cellular Automaton (CA) driven by a Riccati equation that can be solved in a closed form. This property suggests the use of well-founded statistical inference in this nonlinear context. A confirmation of this property is explicitly attained in Guseo and Guidolin (2007b) under a more general framework where a dynamic information network allows the implementation of a dynamic market potential. In this work we address the issue of competition and cooperation among different products or services in a common marketplace. In particular, we introduce a deterministic one–dimensional CA following Boccara (2004) and extend it to a suitable bivariate probabilistic version where an agent may select, at time t, at most one between two competing innovations. This *bivariate* automaton is then simplified under a "mean field approximation", obtaining a continuous representation that gives rise to the Guseo–Bonaldo synchronic duopolistic model, GB–M (see Bonaldo (1991)). Moreover, we study some characterizations of the more complex two–fold *diachronic* case with potential interventions.

2. Multivariate Cellular Automata

A deterministic one–dimensional CA is characterized by three elements: a *population of agents*, Z, a *state function* $s(i,t)$ and a *local evolution rule*, $f(\cdot)$.

The population of agents, Z, corresponds to a set of labels for agents' identification. We assume Z as the set of all integers. The state function $s(i,t) \in Q$ denotes for each agent $i \in Z$ at time $t \in \mathbb{N}^*$ (the set of all positive integers) a level within class $Q = \{0, 1\}$. For agent i, $s(i;t) = 1$ denotes the adoption of a particular innovation. On the contrary, $s(i;t) = 0$ depicts the neutral state. The local *transition rule* is a function $f : Q^{r_\ell + r_r + 1} \to Q$, such that

$$s(i;t+1) = f(s(i-r_\ell;t), s(i-r_\ell+1;t), \cdots, s(i-1+r_r;t), s(i+r_r;t)), \qquad (1)$$

where the integers, r_ℓ, r_r, are the *radii* of the rule.

– Two competing innovations –

We represent now an automaton where an agent at time t may select at most only one between two competing innovations or remain neutral. We may define with an indicator function $I_{(s_1(i;t)+s_2(i;t)=0)}$ if agent i has not adopted at time t where $s_1(i;t)$ and $s_2(i;t)$ denote the state functions related to two different products, innovation 1 and innovation 2,

$$\begin{aligned} s_1(i;t+1) &= s_1(i;t) + [Bi(1,p_1) + Bi(1, q_1\sigma(i;t)]\, I_{(s_1(i;t)+s_2(i;t)=0)} \\ s_2(i;t+1) &= s_2(i;t) + [Bi(1,p_2) + Bi(1, q_2\sigma(i;t)]\, I_{(s_1(i;t)+s_2(i;t)=0)} \end{aligned} \qquad (2)$$

where $\sigma(i;t) = \sum_{n=-\infty}^{\infty}(s_1(i+n;t) + s_2(i+n;t))p(n)$ and $\sum_n p(n) = 1$. In particular, $\sigma(i;t)$ is a common pressure towards adoption based on the knowledge of the product category. Note that if we sum up previous synchronic processes in Equations (2) we obtain an aggregated transition rule, i.e.,

$$s(i;t+1) = s(i;t) + [Bi(1,p) + Bi(1, q\,\sigma(i;t))]\, I_{(s(i;t)=0)}, \qquad (3)$$

where $s(i;t) = s_1(i;t) + s_2(i;t)$ and $p = p_1 + p_2$ and $q = q_1 + q_2$ denote *innovative* and *imitative* components. Moreover, summation among binomial experiments in Equations (2) and (3) follows the *selection rule* (marginalization).

We may approximate the aggregate discrete time system (2) with a continuous representation, under the "mean field approximation" based on $\sigma(i;t) \simeq z(t)/m$, i.e., $z_1'(t) \simeq \sum_i [s_1(i;t+1) - s_1(i;t)]$ and $z_2'(t) \simeq \sum_i [s_2(i;t+1) - s_2(i;t)]$, where, in particular, $z(t) = \sum_i s(i;t) = \sum_i [s_1(i;t) + s_2(i;t)] = z_1(t) + z_2(t)$ is the cumulative aggregate product sales and m the assumed constant market potential. These positions give rise to the differential counterpart, i.e., the Guseo–Bonaldo duopoly model, GB–M.

– Two–fold synchronic competition, GB–M –

The Guseo–Bonaldo equation system, GB–M, depicts a bivariate behaviour of two competing diffusive processes which may have access to a common market. See for instance Bonaldo (1991) and Guseo (2004). This model has been recently and independently rediscoverd by Krishnan et al. (2000). The mechanism which governs the inertial part of imitative components is assumed to be a *common driver*, i.e., a relative common knowledge $z(t)/m$ and is a typical property of a competitive *niche*, a competitive environment or a competitive market. Note that the fraction of common driver, $q_i z(t)/m$, is the specific imitative characteristic of i–th diffusion process. Nevertheless, the joint presence of two competitors gives rise to a non zero competition effect.

Here, we summarize, for simplicity reasons, the duopolystic case but the system may be naturally extended to larger systems of k equations, with $k > 2$,

$$\begin{aligned} z_1'(t) &= m\left(p_1 + q_1\frac{z(t)}{m}\right)\left(1 - \frac{z(t)}{m}\right) \\ z_2'(t) &= m\left(p_2 + q_2\frac{z(t)}{m}\right)\left(1 - \frac{z(t)}{m}\right), \end{aligned} \qquad (4)$$

where $z(t) = z_1(t) + z_2(t)$ depicts the sum of two cumulative diffusion processes, m denotes the limiting state of $z(t)$, as far as $t \to +\infty$, i.e., the assumed constant aggregate carrying capacity or market potential. The integration of System (4) gives rise to

$$\begin{aligned} z_1(t) &= \frac{m(q_1 p - q p_1)}{q^2}\ln\frac{1 + \frac{q}{p}e^{-(p+q)t}}{1 + \frac{q}{p}} + \frac{q_1 m}{q}\frac{1 - e^{-(p+q)t}}{1 + \frac{q}{p}e^{-(p+q)t}} \\ z_2(t) &= \frac{m(q_2 p - q p_2)}{q^2}\ln\frac{1 + \frac{q}{p}e^{-(p+q)t}}{1 + \frac{q}{p}} + \frac{q_2 m}{q}\frac{1 - e^{-(p+q)t}}{1 + \frac{q}{p}e^{-(p+q)t}}. \end{aligned} \qquad (5)$$

Such a solution highlights the competition effect. The parametric functions $(q_1 p - q p_1)$ and $(q_2 p - q p_2)$ are opposite values so that their sum is zero.

– Two–fold diachronic competition, GB–MS –

A modified GB–M that includes a delay c_2 in a second competitor's birth date may be described by the following equation system with new constraints, i.e.,

$$\begin{aligned} z_1'(t) &= m\left(p_1 + q_1\frac{z(t)}{m}\right)\left(1 - \frac{z(t)}{m}\right) \\ z_2'(t - c_2) &= m\left(p_2 + q_2\frac{z(t)}{m}\right)\left(1 - \frac{z(t)}{m}\right) I_{t \ge c_2}. \\ m &= m_1 + m_2 I_{t \ge c_2}, \\ z(t) &= z_1(t) + z_2(t - c_2) I_{t \ge c_2}. \end{aligned} \qquad (6)$$

There is a closed form solution to System (6). Figure 1 and Figure 2 depict the delay effect under different market potential increments.

– Two–fold competition and environmental intervention –

Similar results may be proved under a modified residual market, $\left(1 - \frac{z(t)}{m}\right)x(t)$, with an intervention function $x(t)$ extending to the bivariate case the seminal paper by Bass *et al.* (1994).

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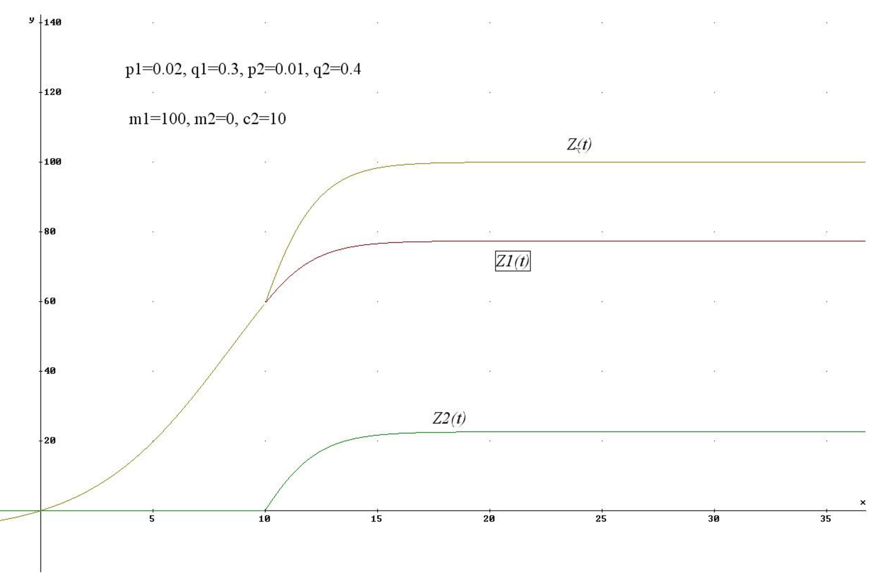

Figure 1: *Two–fold diachronic competition, GB–MS:* $m_2 = 0$, $p_1 = 0.02$, $p_2 = 0.01$.

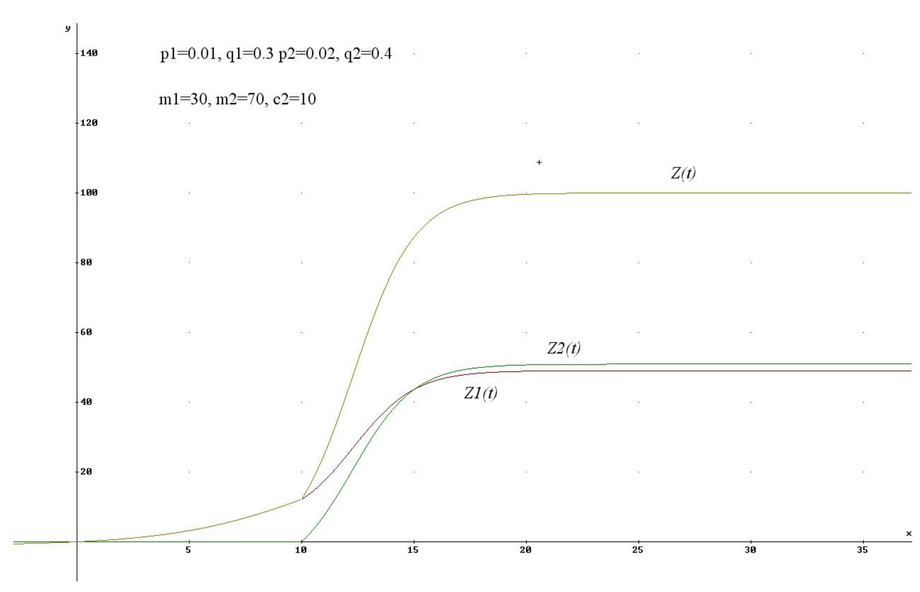

Figure 2: *Two–fold diachronic competition, GB–MS:* $m > m_1$, $p_1 = 0.01$, $p_2 = 0.02$.

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