Cellular Automata with Network Incubation Period versus Perturbed Riccati Equation Models in Information Technology Innovation Diffusion

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ABSTRACT. Innovation diffusion of network goods determines direct network externalities that exhibit delayed full benefits depressing sales for long periods. We model a multiplicative dynamic potential market driven by a latent heterogeneous individual threshold derived from a basic economic theory which is embedded in a special Cellular Automata representation. The corresponding mean field approximation of its aggregate version is a Riccati equation with a closed form solution that allows a separation between an incubation period and a subsequent take–off due to a sufficient critical mass. Weighted nonlinear least squares methodology is the main inferential technique. An application is analyzed with reference to USA fax machines diffusion.

1 Introduction

A central aspect of innovation diffusion is the physical and technological character of good or service. Normal or "stand alone" goods create benefits for a buyer not depending on the status of other potential or actual buyers. In this case we may argue that potential market is, within natural limits, essentially a stationary process over its life cycle.

Network industries, i.e., telephony, electricity, natural gas, roadways, internet, railroads, facsimiles, etc., define expanding physical networks. The increase of sales of network goods (network terminals or network services) determines external benefits for all network participants because this positively modifies the utility function of active clients. Nevertheless, individual utility is a latent function: we observe only decisions, i.e., adoptions. Such external benefits are not market mediated and are usually termed as "direct network externalities".

Network goods exhibit delayed full benefits so that sales are depressed for long periods and successful industries with network externalities highlight *positive critical mass*. For successful network goods we may argue that potential market is, without minor details, characterized by a local depression which describes a *change point* time, $\hat{t}$, separating two different regimes: a long preliminary incubation period followed by a sudden "take–off" that is associated with the attainment of a *positive critical mass*.

The issue of network externalities was modelled in the past by many authors in different areas. In economics, we mention, among others, Economides and Himmelberg (1995). Within marketing science there are interesting integrated contributions by Goldenberg *et al.*(2005). Within sociological and statistical sciences there are interesting advances that emphasize some multiphase aspects of diffusion under network effects, Mahler and Rogers (1999), Seber and Wild (1989).

In Complex System Analysis some important references among others are, Moldovan and Goldenberg (2004), Boccara (2004). Cellular Automata (CA) are special models within Complex Systems theories. For some oriented references see, for instance Boccara and Fukś (1999), Guseo and Guidolin (2007).

In order to describe the chilling effect of network externalities we suggest in Section 3 a general model which is based on Cellular Automata representation describing an adoption process within a dynamic potential market driven by heterogeneous individual threshold coherently with basic economic theory (see, Section 2). We next transform Cellular Automata representation, Section 3, in an aggregate form by a *mean field approximation* that allows a joint differential description of adoption process nested within a suitable dynamic potential market under network externalities effects and, in particular, under critical mass effects.

The proposed differential representation, a Riccati equation, gives rise to a closed form solution that allows a simple detection of critical time, $\hat{t}$, separating incubation period from take–off and corresponding critical mass. Some statistical features are involved within inferential process based on contiguous different regimes. In particular, special weighting is essential within NLS (Nonlinear Least Squares) techniques. Section 4 examines a direct application to USA fax machines time series.In particular, some evidence is given to estimated latent potential market by recognizing its evolution between two sequential regimes: the incubation period and the successive take–off due to the attainment of a critical mass.

2 ECONOMIC AND SOCIAL MODELLING: A COMMON KEY RULE

Economides and Himmelberg (1995) assume that consumers expect a normalized network size ν^e such that $0 \leq \nu^e \leq 1$ and define a *network externalities function* which describes a common perceived value of the good,

$$v(\nu^e) = k + \delta f(\nu^e), \tag{1}$$

where k gives the *value of the good* in absence of network effects, δ is an indicator function taking the value 1 if there are network externalities and zero elsewhere. Function $f(\cdot)$ is monotone increasing with initial condition $f(0) = 0$, $f'(\cdot) > 0$ and $f''(\cdot) \leq 0$ in order to describe an increasing utility under larger expected sizes of networks.

The Authors assume a special definition of the *willingness to pay* for one unit of the good in a network of expected size ν^e, i.e., $w(h,\nu^e) = hv(\nu^e)$, where $h \in [0,1]$ is a penalizing index characterizing a specific consumer and $P(h)$ is a corresponding cumulative distribution function. This assumption differs from a more common additive one (see, for instance, Katz and Shapiro (1985)) under which all consumers receive the same benefit from the same network.

Given expectations ν^e and price p, they define the index h^* of marginal consumer as a solution of Equation $p = w(h,\nu^e)$ so that, in particular, $h^* = p/v(\nu^e)$. Under given expectations and price all consumers with index $h \geq h^*$ buy the good so that the network size at price p is $n = 1 - P(h^*)$, defining the demand for the network good so that they can write the willingness to pay for the last consumer in a network of size n with expectations ν^e, i.e., $p(n,\nu^e) = v(\nu^e)P^{-1}(1-n)$. Previous equation is a basic tool for a systematic discussion of a critical mass under different market structure hypotheses.

We propose a different definition of customer index h focusing on the marginal consumer in a dynamic way at time t so that what matters is the personal evaluation of the ratio $h = p/\tilde{v}(\nu)$. This ratio may be interpreted as a random variable H representing a dependent individual assessment where price p is of public domain, $\tilde{v}(\nu)$ may be considered as a more complex function than (1) based on current observed normalized market $\nu(t) = y(t)/m(t)$ with $y(t)$ the observed absolute cumulative adoptions and $m(t)$ the absolute potential market, the expectations $\nu^e(t)$, the personal value $\tilde{k}$ of the good which may be different among current potential consumers and other personal assessment components. Following our proposal, we argue that $h \in H$ is a plausible individual expression of divergence between current price p and a personal evaluation of the "value of the good". This ratio is a dimensionless pure number. A consumer may become a potential buyer if his ratio h is lower than the current observed normalized expressed preference $\nu(t)$. Accordingly, we define the current normalized potential market with a probability $P(H \leq \nu(t))$ and the absolute *dynamic potential* of the system as

$$m(t) = U\,E\left(I_{(H\leq\nu(t))}\right) = U\,P\left(H \leq \nu(t)\right), \tag{2}$$

where U is the asymptotic market potential and $I_{(H\leq\nu(t))}$ is an indicator function acting on random variable H.

We can study the behaviour of threshold effect under a normality assumption, i.e.,

$$m(t) = U\,P\left(H \leq \nu(t)\right) = U\,\Phi\left[(\nu(t)-\mu)/\sigma\right], \tag{3}$$

and model the mean μ in order to avoid a stationarity assumption of threshold distribution during the *incubation period*. A flexible alternative in mistrust description is based on a polynomial function, $\mu(t) = a + bt + ct^2$, that may be used only within a limited incubation period and, under particular circumstances, it may be unexpectedly increasing.

The definition of *critical mass* in our approach is an operational notion related to the joint combination of dynamic potential market and actual adoption process. It is the level in cumulative sales that corresponds to a suitable change point $\hat{t}$ that separates strictly incubation period from a sudden take–off. The technical identification of $\hat{t}$ is based on a local minimum of $P(H \leq \nu(t))$.

3 Network Incubation Period in a Cellular Automaton

Here we follow, only partially, some CA notations expressed in Boccara and Fukś (1999). In particular, we use a univariate coordinate system, $i \in W = \{1, 2, ..., U\}$, where i is a generic adopting unit whose *state* at time t is an indicator function $s(i;t)$.

The *transition rule* of a CA must be properly specified in order to recover possible effects of an *incubation period* due to special *network externalities* that affect diffusion.

We suppose that the change of state of i–th unit is driven by two major processes: the first one refers to innovative and imitative contribution to a *perturbed* residual market penalized by a threshold or resistance effect, the second one is a neighborhood effect due to the local relative variation of dynamic potential market,

$$s(i;t+1) = s(i;t) + I_{(h_i\leq\nu(t))} \cdot A\,\frac{x(t)}{m(t)} I_{(s(i;t)=0)} + s(i;t) \cdot \frac{m'(t)}{m(t)}. \tag{4}$$

where $h_i \in H$ is the individual threshold for susceptibility and $\nu(t)$ represents the density of the adoption process or the mean temporal unitary fraction of adoptions at time t. If the individual threshold is lower than $\nu(t)$, then the indicator function $I_{(h_i \leq \nu(t))}$ is set to 1 depicting the admissibility of the *inductive experiments*. Binomial inductive experiment $A = [Bi(1, p) + Bi(1, q\nu(t))]$ represents, respectively, the contributions of innovative and imitative components. Integrable function $x(t)$ allows a modification of perceived residual market, $(m(t) - y(t))$, and dynamically represents environmental pressure variations, economic choices, political regulations or firms strategies effects. Its equilibrium value is 1. This factor is very important as a contrasting tool against the negative effects of *network externalities* in delaying adoptions. The component $s(i;t).\frac{m'(t)}{m(t)}$ in Equation (4) describes an infinitesimal variational contribution to the individual state due to the varying effect of carrying capacity over time.

The average behaviour of Equation (4) followed by a summation over all cell states $s(i;t)$, $i \in W$ is an aggregate discrete time *cumulative evolutive model* under a mean field approximation, i.e.,

$$y(t+1) = y(t) + m(t)\left\{\left(p + q\frac{y(t)}{m(t)}\right)\frac{(m(t) - y(t))}{m(t)}x(t)\right\} + y(t)\frac{m'(t)}{m(t)}. \tag{5}$$

With a continuous approximation of Equation (5) and position $\nu(t) = y(t)/m(t)$, we attain $\nu' = (p + q\nu)(1 - \nu)x(t)$. The general solution of previous equation is presented in Guseo and Guidolin (2007) (Appendix), i.e.,

$$y(t) = m(t) \cdot \nu(t) = U\,P(H \leq \nu(t)) \cdot \nu(t), \tag{6}$$

where,

$$\nu(t) = \frac{1 - e^{-(p+q)\int_0^t x(\tau)d\tau}}{1 + \frac{q}{p}e^{-(p+q)\int_0^t x(\tau)d\tau}}. \tag{7}$$

The explicit closed form solution, as expressed by Equations (6 – 7) is an extension of generalized Bass model (GBM) with a variable potential (see, for instance, Guseo (2004) and Bass *et al.*(1994)). The simultaneous description of network externalities effects that affects dynamic potential (market) and the formal presence of an explicit intervention function $x(t)$ that allows the measurability of the effects of convenient strategic marketing or management contrasting actions, are strength elements of the present class of models. Note that, by examining Equations (6 – 7), the exponent $\int_0^t x(\tau)d\tau$ modifies with compensative actions the geometry of time and not the asymptotic potential U.

4 USA Fax Machines: network externalities vs marketing efforts

In order to examine the performance of model defined by Equations (6 – 7) we consider the well–known example of USA fax machines (1965–94). The original source is CBEMA (1998) (*Information Technology Industry Data Book*). This particular series starts in the mid 1960s with a long incubation period, sometimes called "chilling effect" (see Goldenberg *et al.*(2005), that lasts for about twenty years followed by a fast *take–off*. This long incubation period is quite rare since the majority of durables and common commodities have a shortened life cycle today so that sometimes the presence in the series of significant incubation periods preceding the takeoff is not so easy to identify statistically.

A good reason for a long left tail in USA fax machines evolution is the network externalities effect balanced with high price/quality ratios. Nevertheless, we suppose that such a device was supported with a long and efficient marketing and management effort by producing firms under evolving technologies. Our aim is the detection of the presence and weight in the USA

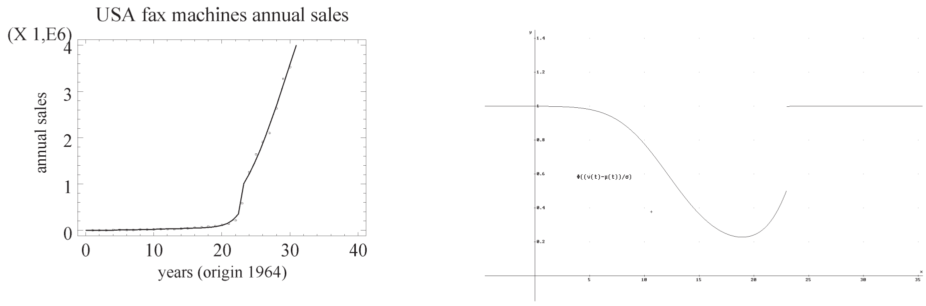

Figure 1. (a) Usa Fax Machines.Bass Model and Network externalities: normal linear threshold and weighted NLS; (b) normal threshold probability linear evolution.

fax machine time series of both partially compensating effects. We assume a Norton and Bass (1987) perspective of USA fax machines evolution by describing non cumulative sales as a change of regime from a lower level (zero) to an upper stationary level. In other words we suppose that the market will reach a stable level in the fax machines stock with an equilibrium park not affected by successive absorbing generations. Previous hypothesis may have sense in a suitable right neighborhood of observed data. We consider as a first step a simple modelling of network effects with no intervention function under normal threshold distribution affecting dynamic potential market with a constant μ parameter, or a linear evolution of $\mu(t)$ within a particular temporal horizon. Preliminary results were unsatisfactory under Equations (6) and (7) because standard NLS (Nonlinear Least Squares) inference tools were used. In particular, we observed a systematic zero estimate (an evident underestimation) during all incubation period. The statistical reasons of such a lack of local identification may be recognized in the structural difference between the two subsequent regimes and may be overcome by two strong modifications of inference: 1) a suitable weighting in NLS procedure. We found a satisfactory performance under a simple weighting function, i.e., $w(t) = 1/y(t)$ and, 2) a limited action of threshold distribution during incubation period . We found a good performance of the model by assuming an extended incubation period between 1965 and 1987 and a linear behaviour for $\mu(t)$ representation under normal threshold distribution. The estimation results are summarized in Table 1. We note a good global fitting, $R_1^2 = 0.994979$, with a good perfor-

Table 1. USA fax machines diffusion. Parameters estimates of a Bass model with network externalities: normal linear threshold and weighted NLS. ( ) marginal linearized asymptotic 95% confidence limits

U	p	q	a	b	s	R_1^2	$D-W$
6464290	0.0000773	0.28078	-0.17579	0.01409	0.051295	0.994979	2.0046
(4684260)	(0.0000410)	(0.24827)	(-0.27661)	(0.00816)	(0.022734)	SSE :	
(8244320)	(0.0001135)	(0.31330)	(-0.07496)	(0.02004)	(0.079856)	[88498]	

mance in linearized asymptotic marginal confidence intervals. The graphical representations of both annual sales and Bass model under network externalities, with an assumed average linear threshold and weighted NLS are depicted in Figure 1 (a). We note, in particular, that the chilling effect during incubation period is properly estimated.

Under previous results we can examine some details in potential market evolution $m(t) = U\Phi((\nu(t) - \mu(t))/\sigma)$ with a linear hypothesis in $\mu(t)$ evolution. In Figure 1 (b) we represent, for USA fax machines, the normal threshold probability average linear evolution under a Bass model with network externalities: $\Phi((\nu(t) - \mu(t))/\sigma)$. Such a probability depicts a non monotone behaviour of receptiveness to innovation during the incubation period. In particular, network externalities increase resistance, with a decreasing probability, from 1965 till 1983 (1964+19) where a minimum in $\Phi(\cdot)$ is attained for $\hat{t} = 19$. Such a time $\hat{t}$ may denote the temporal origin of the "social awareness" due to a perceived *critical mass* presence of the new technology. Starting from 1983 the normalized potential, $\Phi(\cdot)$, rapidly emerges to its stationary level (complete receptiveness) in few years, 1987. The critical mass evaluated at 1983 is about 550000 fax machines.

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