# Cellular Automata and Riccati Equation Models for Diffusion of Innovations

*Automi Cellulari e l'Equazione di Riccati nei Modelli di Diffusione delle Innovazioni*

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**Riassunto:** La diffusione di innovazioni in un sistema sociale è stata studiata in settori scientifici differenti, avendo spesso come riferimento il lavoro pionieristico di Bass (1969) fondato su un approccio aggregato differenziale. Tuttavia, il fatto che i canali di comunicazione fra persone possano avere un raggio d'azione limitato ha stimolato la modellazione di Automi Cellulari (CA). Qui si propone un'estensione del CA di Boccara e Fuks (1997) ricorrendo inoltre ad un'approssimazione continua della corrispondente equazione a tempo discreto. Il modello approssimato rientra in un'equazione di Riccati non autonoma che incorpora una funzione utile nella rappresentazione di interventi esogeni e nel controllo del loro impatto. I risultati principali si riferiscono alla soluzione in forma chiusa, all'analisi per l'identificazione e la stima e all'applicazione statistica.

**Keywords:** diffusion process, cellular automaton, Riccati equation, generalized Bass model, automata network.

## 1. Introduction

The Bass (1969) model, BM, and its extensions represent valid tools for modeling and forecasting diffusion of innovations processes. One of the main advantages connected with this class of models is the opportunity of working with aggregate data, which is generally simpler and less expensive than trying to recover data at the individual level. Using aggregate models for the analysis of diffusion processes involves some reduction of phenomena. However, it allows a very reliable description of them, through the estimation of few parameters. The forecasting ability of aggregate models is linked to their simplicity and parametric flexibility. The standard BM is based, for instance, upon two simple sub–populations, *innovators* and *imitators*, that are latent categories because actual stock data, within finite time intervals, are not separated. Nevertheless, the poor attention paid to the individual level has been perceived as a concrete problem especially in the economic context. In order to recognize different *local* diffusions, Cellular Automata models (CA) are a recent example of the trial to fill in this shortage. Boccara and Fuks (1997) and Boccara (2004) have proposed an interesting representation of special CA. In Section 2 we briefly introduce a class of CA and, in particular, we approximate a natural extension (BFG) of the Boccara and Fuks (1997) model with a continuous Riccati representation of the corresponding discrete time equation. In Section 3 we solve, in closed form, a quite general non autonomous Riccati equation. In Section 4 we apply previous results to BFG and examine statistical aspects concerned with inference and applications.

## 2. Cellular Automata Models: an Extension

A CA is composed of a grid of cells and each of them can assume a specific state (e.g. neutral, adopter), given a finite number of $k$ possible states (for example two). It is assumed (see Boccara and Fuks 1997) that the state of the cell $i \in Z$ (the set of all integers) at time $t+1$, $s(i, t+1)$, depends on the state $s(i, t)$ at time $t$ and on a specific range $\sigma(i, t)$, defined as a kind of *pressure* on cell $i$ to adopt,

$$\sigma(i,t) = \sum_{n=-\infty}^{\infty} s(i+n,t)p(n), \tag{1}$$

where $p(n)$ is a probability distribution. Moreover, an individual (cell) may become an adopter (i.e. from the state zero to the state one) with a probability that is proportional to $\sigma(i, t)$, $P_{1\leftarrow 0} = q\sigma(i, t)$, and the decision to adopt is reversible with a probability $r$, $P_{0\leftarrow 1} = r$. Defining with $\rho(t)$ the average density of adopters at time $t$ and with $\eta(t) = 1 - \rho(t)$ the average density of neutrals, where $\langle \ \rangle$ denotes a spatial average, we have that at time $t+1$

$$\begin{pmatrix} \eta(t+1) \\ \rho(t+1) \end{pmatrix} = \begin{pmatrix} \langle P_{0\leftarrow 0}\rangle & \langle P_{0\leftarrow 1}\rangle \\ \langle P_{1\leftarrow 0}\rangle & \langle P_{1\leftarrow 1}\rangle \end{pmatrix} \cdot \begin{pmatrix} \eta(t) \\ \rho(t) \end{pmatrix}, \tag{2}$$

and, therefore,

$$\begin{pmatrix} \eta(t+1) \\ \rho(t+1) \end{pmatrix} = \begin{pmatrix} 1 - q\langle\sigma(i,t)\rangle & r \\ q\langle\sigma(i,t)\rangle & 1-r \end{pmatrix} \cdot \begin{pmatrix} \eta(t) \\ \rho(t) \end{pmatrix}. \tag{3}$$

Here we propose an extension, say BFG, of the Boccara and Fuks (1997) CA in which we change the probability of becoming an adopter from a logistic pattern $P_{1\leftarrow 0} = q\langle\sigma(i, t)\rangle$ to a Riccati–like framework in order to recover the initializing aspects of the diffusion which can not be omitted and are included in Bass family models, i.e.,

$$P_{1\leftarrow 0} = p + q\langle\sigma(i,t)\rangle. \tag{4}$$

Furthermore, we approximate the finite difference $\rho(t+1)-\rho(t)$ with the prime derivative, $\rho'(t)$, and refer to $\rho(t)$ as a limiting behaviour of $\langle\sigma(i, t)\rangle$ for a spreading distribution $p(n)$. The proposed limiting continuous time BFG model is, therefore,

$$\rho'(t) = -r\rho(t) + (p + q\rho(t))(1 - \rho(t)). \tag{5}$$

Note that Equation (5) is a representation of a possible weaker assumption in the spatial memory depth of the imitative pattern followed by Boccara and Fuks (1997) modified model, i.e., $\rho'(t) = -r\rho(t) + (p + Q\langle\sigma(i, t)\rangle)(1 - \rho(t))$. Here we may assume $\langle\sigma(i, t)\rangle$ as a discounted function of $\rho(t)$, i.e. $\langle\sigma(i, t)\rangle = s\rho(t)$ so that $Q\langle\sigma(i, t)\rangle = Qs\rho(t) = q\rho(t)$. Parameter $Q$ denotes the pure imitative effect, and $s$ represents a normalized spatial memory depth, i.e., a share of neighboring pressure $\sigma(i, t)$. Nevertheless, Equation (5) can not identify separately $Q$ and $s$. However, this is a minor problem in prediction because parameter $q$ jointly summarizes the pure imitative component and a simple version of local communication pattern.

## 3. A Non Autonomous Riccati Equation

In order to solve Equation (5) let us consider a non autonomous Riccati equation

$$y' + (ay^2 + by + c)x(t) = 0, \quad a, b, c \in R, \quad I(t) = \int_0^t x(\tau)d\tau < \infty, \tag{6}$$

where $x(t)$ is an integrable function, $r_i = (-b \pm \sqrt{b^2 - 4ac})/2a \in R, i = 1, 2$, with $D = a(r_2 - r_1) = \sqrt{b^2 - 4ac} > 0$ and $r_2 > r_1$ so that an equivalent representation is $y' + a(y - r_1)(y - r_2)x(t) = 0$. Let us denote $\dot{y} = y - r_2$ and $\dot{y}' = y'$ with initial conditions $y(0) = 0$ or $\dot{y}(0) = -r_2$ then previous equation divided by $\dot{y}^2$ is equivalent to $\frac{\dot{y}'}{\dot{y}^2} + \{a(r_2 - r_1)\frac{1}{\dot{y}} + a\}x(t) = 0$. With a new variable change, $\hat{y} = \frac{1}{\dot{y}}$, so that $\hat{y}' = -\frac{\dot{y}'}{\dot{y}^2}$ and initial condition $\hat{y}(0) = -\frac{1}{r_2}$ we obtain equation

$$\hat{y}' = \{a(r_2 - r_1)\hat{y} + a\}x(t)\,, \tag{7}$$

which may be integrated following, for example, Apostol (1978 p. 31).

Let us denote $G(t) = e^{a(r_2 - r_1)\int_0^t x(\tau)d\tau}$ so that the solution of equation (7) is

$$\begin{aligned}
\hat{y} &= -\frac{1}{r_2}G(t) + G(t)a\int_0^t x(\tau)e^{-a(r_2-r_1)\int_0^\tau x(\xi)d\xi}d\tau \\
&= -\frac{1}{r_2}G(t) + G(t)a\left[-\frac{1}{a(r_2 - r_1)}e^{-a(r_2-r_1)\int_0^t x(\xi)d\xi} + \frac{1}{a(r_2 - r_1)}\right] \\
&= \frac{G(t)r_1 - r_2}{(r_2 - r_1)r_2}.
\end{aligned} \tag{8}$$

In terms of the initial variable, $y = \frac{1}{\hat{y}} + r_2$, we obtain

$$y(t) = r_2 + \frac{r_2(r_2 - r_1)}{G(t)r_1 - r_2} = \frac{1 - G^{-1}(t)}{\frac{1}{r_2} - \frac{1}{r_1}G^{-1}(t)} = \frac{1 - e^{-a(r_2-r_1)\int_0^t x(\tau)d\tau}}{\frac{1}{r_2} - \frac{1}{r_1}e^{-a(r_2-r_1)\int_0^t x(\tau)d\tau}}. \tag{9}$$

If $\lim_{t\to\infty} I(t) = +\infty$, we attain a limiting behaviour of $y(t)$, i.e., $\lim_{t\to\infty} y(t) = r_2$.

## 4. Statistical Modeling and Discussion

The proposed extension of Boccara and Fuks (1997) model, BFG, by Equation (5) in Section 2 may be represented within previous Riccati equation in order to incorporate interventions by function $x(t)$, i.e.,

$$y' = \{-ry + (p + qy)(1 - y)\}x(t), \quad r, p, q, \in R,\ 0 < p < q,\ 0 < r. \tag{10}$$

The extended Boccara-Fuks model, BFG, is a special case under $x(t) = 1$ and the well-known Generalized Bass model, GBM, by Bass *et al.* (1994) is attained for $r = 0$. Standard Bass model, BM, by Bass (1969) includes both the constraints, $x(t) = 1$ and $r = 0$.

The factorization of Equation (10) is

$$y' + q(y - r_1)(y - r_2)x(t) = 0, \tag{11}$$

with real roots, for $D = \sqrt{(r+p-q)^2 + 4pq} > 0$, equal to

$$r_i = \frac{-(r+p-q) \pm D}{2q}, \quad i = 1, 2, \tag{12}$$

so that we have $a(r_2 - r1) = D$ and the closed form solution is

$$y(t) = \frac{1 - e^{-D \int_0^t x(\tau) d\tau}}{\frac{1}{r_2} - \frac{1}{r_1} e^{-D \int_0^t x(\tau) d\tau}}. \tag{13}$$

The absolute scale representation of natural diffusion may be obtained in a straightforward manner, $z(t) = My(t)$, so that the asymptotic behaviour is $m = \lim_{t \to +\infty} z(t) = Mr_2$.

The perturbed closed form solution is very useful for a statistical approach to forecasting and simulations based upon the detection of the internal rules that generate a CA under a widespread distribution of local influence on individual adoption or withdrawal of an innovation. The statistical implementation of model (13) may adopt different error structures. In a nonlinear regressive approach we consider a particular model for observations, $w(t) = z(t) + \varepsilon(t)$, with an i.i.d. residual $\varepsilon(t)$. A useful complementary approach is based on ARMAX representation with a standard nonlinear estimation as a first step (see e.g. Guseo (2004), Guseo and Dalla Valle (2005) and Guseo *et al.* (2005)). Some applications of model (13) refer to the analysis of italian truck tractors stock (1950 – 2002) and to the weekly diffusion of a bank–account suitable for north-east firms of Italy.

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