Worldwide Cheap and Heavy Oil Productions: A Long-Term Energy Model☆

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Abstract

Crude oil, natural gas liquids, heavy oils, deepwater oils, and polar oils are non-renewable energy resources with increasing extraction costs. Two major definitions emerge: regular or 'cheap' oil and non-conventional or 'heavy' oil. Peaking time in conventional oil production has been a recent focus of debate. For two decades, non-conventional oils have been mixed with regular crude oil. Peaking time estimation, and the rate at which production may be expected to decline, following the peak, are more difficult to determine. We propose a two-wave model for world oil production pattern and forecasting, based on the diffusion of innovation theories: a sequential multi-Bass model. Historical well-known shocks are confirmed, and new peaking times for crude oil and mixed oil are determined with corresponding depletion rates. In the final section, possible ties between the dynamics of oil extraction and refining capacities are discussed as a predictive symptom of an imminent mixed oil peak in 2016.

Keywords: oil depletion, diffusion process, sequential multi-Bass model

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1. Introduction and a review

Oil is a non-renewable resource formed in the geological past, in a process that took millions of years. It is not a homogeneous resource and, for this

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reason, we face different classification systems adopted by the Energy Information Administration (EIA), International Energy Agency (IEA), *Oil and Gas Journal* (OGJ), *World Oil Magazine* (WOm), *British Petroleum Statistical Review of World Energy* (BP), *World Oil and Gas Review* (WOeni), etc. Two major definitions emerge: regular or 'cheap' oil and non-conventional or "heavy" oil. The former refers to light and sweet crude oils, while the latter brings together different resources: natural gas liquids (NGL), heavy oils like tar sands and oil shales, deepwater oils, and polar oils, all of which incur increasing costs. The contribution of non-conventional oil in supplying the economy was very limited between 1900 and the 1970s. See, for instance, Campbell and Laherrère (1998). Campbell argues an asymptotic scenario with a non-conventional oil share around one-fifth.

Recent advances in producing and measuring techniques such as Q-technologies, new streamer technologies, periscope drilling, seismic-guided drilling, and deepwater technologies by Schlumberger (see Gould 2009), constitute a strong example of the effort spent in order to overcome today's recession in the non-renewable energy area, the actual technological driver of worldwide economy. The *Deepwater Horizon* accident in 2010 is a paradigmatic example of the increasing risks and related direct and indirect costs in non-conventional situations.

Production at given oilfields is determined by common stimuli, such as transport costs, regulatory pressures, nearby markets, etc. Measuring oil in the ground is a difficult task because technical, strategic, and economic viewpoints are partially conflicting.

The scientific debate about the new challenges concerning today's energy crisis is a focal point in many research areas. These research areas may contribute to overcoming the emerging difficulties in the oil industry through the strengthening of viable, sustainable, and socially acceptable new technologies. A recent comprehensive report, produced by the Technology and Policy Assessment function of the UK Energy Research Centre, UKERC (2009), examines about 500 international papers concerning the global oil depletion issue. It is an assessment of the evidence for a near-term peak in global oil production. Different theories and methodologies are examined with reference to the definition and estimation of 'reserves.' Two main approaches are highlighted. A 'realistic' vision considers the dynamics of extraction as a function of direct and indirect costs, energy return on energy investment (EROEI), strategic opportunities, and environmental constraints. An 'optimistic' approach assumes a future accessibility of oil resources, not relevant for today's economy, with the employment of new sophisticated technologies. For an introduction to the debate between realistic and optimistic approaches concerning crude oil perspectives see, for instance, Maugeri (2010) and Zecca et al. (2010). Within the optimistic approach, Greene et al. (2006) focus on the description of a transition to unconventional oil resources. They use a simulative risk model in order to combine alternative world energy scenarios from IIASA, USGS, and IEA studies. In their approach, the issue is not framed as a peaking time question for conventional oil, but in terms of a transition to unconventional oil resources.

The aim of the present paper is to combine the qualitative rationale expressed in Greene et al. (2006) with a different methodological perspective, based on diffusion of innovation theories (see, for instance, Guseo et al. 2007).

The main focus of this paper is devoted to oil reserves estimation through the characterisation of the *evolutionary production pattern* and statistical *Ultimate Recoverable Resource* (URR) determination. The URR for oil is the 'total amount of a finite resource which may be obtained at the end of extraction or production process as a result of all concurring forces.' For this reason, it must be jointly estimated from production evolution and not as a separate parameter as frequently observed in Hubbert's methodologies. This asymmetry, often assumed in a part of literature, may be questionable for a possible lack of motivation. See, among others, Mohr and Evans (2007, 2009) and Mohr (2010). Historical *production data* summarise the variable joint contributions of technological, economic, and social effects, including dynamic learning, on the production of a finite resource.

The statistical and forecasting literature on *Ultimate Recoverable Resource* (URR) estimation is quite limited, with some important exceptions. Two reviews in this area are those by Adelman and Jacoby (1979), and by Kaufmann (1988). Recent econometric extensions of the well-known logistic Hubbert model are provided by Kaufmann (1991), Cleveland and Kaufmann (1991), Pesaran and Samiei (1995), and Berg and Korte (2008). Oil aggregate demand is strongly correlated with the diffusion of the corresponding technologies in transport, heating appliances, electric power generation, etc. For these reasons, extraction data may be interpreted within a *diffusion of innovation* framework under commonly observed exogenous interventions that modify diffusion trajectories (see the generalised Bass model in Bass et al. 1994). Under a *finite life cycle* hypothesis, the unknown asymptotic *market potential* in a quantitative marketing context interprets the role of URR in the case of oil. Reserves are then indirectly obtained as a simple difference between estimated URR, through historical production data and actual cumulative production, avoiding overestimation effects due to 'financial reasons.' This approach, generalising the Hubbert one, was expressed in Guseo and Dalla Valle (2005) and in Guseo et al. (2007), where the Generalised Bass Model (GBM, see Bass et al. 1994), is the main tool in estimating locally perturbed extraction processes.

However, many systems exhibit complex dynamics that cannot be reduced to a single life cycle. We often observe multiple processes that may occur *simultaneously* or, more frequently, *sequentially.* The bi-logistic growth modelling by Meyer (1994) and the subsequent primer for the Loglet Lab Software by Meyer et al. (1999) may represent a first general approach in the analysis of simultaneous growth processes, based on the pioneering paper by Marchetti (1980) among others. We emphasise the simultaneous nature of the proposed bi-logistic or multi-logistic models, denoted by the subscript ML, for an implicit mathematical constraint that characterises the logistic framework components. As a matter of fact, each local distribution is a three parameter positive 'density' for $-\infty < t < +\infty$, with *positive* initial condition at $t = 0$,

$$
{}_{ML}z'(t) = \sum_{i=1}^{s} \frac{m_i r_i e^{-r_i(t-t_{pi})}}{\left(1+e^{-r_i(t-t_{pi})}\right)^2}, \qquad -\infty < t < +\infty, \tag{1}
$$

where ${}_{ML}z'(t)$ denotes the mixed instantaneous production at time t, and for each cycle $i = 1, 2, \cdots, s$ we have a carrying capacity m_i, r_i is a 'steepness rate' or 'slope' and t_{pi} is the peak time of the unimodal symmetric local function. Notice that the local peak value, $P_i = m_i r_i/4$, is obtained for $t = t_{pi}$. Model (1) assumes that the s logistic processes are always active for each time t. This is surprising and contrary to intuition in the case of sequentially active (production) processes. This kind of modelling was firstly applied, in the energy context, by Al-Fattah and Startzman (2000). For a recent study concerning world oil production, see the iterated application to the analysis of production forecasts for the 47 most influential countries in terms of URR by Nashawi et al. (2010). A further application of the multi-logistic approach, denoted in the energy context as 'multi-Hubbert model' was presented in Imam et al. (2004) with reference to natural gas extraction and peaking times.

A similar approach in multi-processes modelling was recently proposed by Maggio and Cacciola (2009). The 'kernel' function, depicting local growth in production, is expressed through a special extension of the logistic (or Hubbert) traditional model, denoted by the subscript L, through trigonometric hyperbolic functions, namely,

$$_{L}z_i{}'(t) = \frac{m_i r_i e^{-r_i(t-t_{pi})}}{\left(1+e^{-r_i(t-t_{pi})}\right)^2} = \frac{2P_i}{(1+\cosh(r_i(t-t_{pi})))}, \tag{2}$$

by introducing in the mixed instantaneous production model at time t, $_{ML}\tilde{z}\,'(t)$, a shape parameter $0 < k_i < 1$ within each cycle, that symmetrically augments (for $k_i << 1$) the variability of local distribution of extractions over time,

$$_{ML}\tilde{z}\,'(t) = \sum_{i=1}^{s} \frac{2P_i}{(1+k_i\cosh(r_i(t-t_{pi})))}. \tag{3}$$

This interesting extension does not have a dual simple interpretable differential equation as in the pure logistic case. See, in particular, the corresponding formal representation discussed in Appendix A.

A first crucial aspect in modelling interpretable deviations from a standard S-shaped growth must describe local or significant interventions which do not rise from the internal dynamics of the cycle but depend upon external factors (political, technological, strategic, economic, etc.). A second crucial aspect is related to the previous surprising constraint induced by the logistic model with positive initial conditions. Logistic function is mathematically defined over the complete real line with a positive initial condition at time $t = 0$, where $_{L}z_i{}'(0) = m_i r_i e^{-r_i t_{pi}}/(1+e^{-r_i t_{pi}})^2$, admitting a non-negative production contradictory preceding the observed process. A third crucial aspect is a possible confounding effect, induced by an over-parameterised multi-logistic modelling, that may erroneously exchange an exogenous intervention, for a new cycle for which no plausible interpretation is allowable.

For at least two decades in publicly reported data, non-conventional (heavy and sour) oils have usually been mixed with regular (light and sweet) crude oil, introducing uncertainties due to the aggregate behaviour of different resources. The emerging question is how to reconstruct component dynamics starting from an aggregate time series of global ‘oil’ production. Peaking time estimation and the rate at which production may be expected to decline following the peak are more difficult to determine.

In Section 2 we propose a two-wave model, namely a sequential multi-Bass model, for world oil production patterns and forecasting. Based on the diffusion of innovation theories, it gives a positive solution to the above-mentioned crucial problems occurring in a pure multi-logistic approach. In particular, the proposed two-wave model expresses correctly the sequential nature of the observed extraction processes and their partial overlapping. Moreover, the imputation of observed large deviations, due to external factors, is much more pertinent, and avoids confounding them with new emerging cycles that are quite rare in this context. In Section 3 we compare the dynamics of mixed oil production with the corresponding refining capacity in order to understand possible ties and predictive symptoms. Section 4 is devoted to some conclusions.

2. A two-wave model for global oil production and forecasting

The rate adoption curve of the Bass model, (BM), ${}_1z'(t) = (p+q\,{}_1z(t)/m)$ $(m-{}_1z(t))$, for $0 \leq t < +\infty$ and initial condition ${}_1z(0) = 0$, depicts a baseline evolution: a distribution that includes, for $p \to 0$, the logistic function (Verhulst, 1838), and, therefore, Hubbert's model. Function ${}_1z'(t)$ denotes the instantaneous sales at time t and ${}_1z(t)$ represents the cumulative sales. The parameter m represents the unknown market potential, while p and q denote the *innovative* and *imitative* evolutionary unknown parameters respectively. The Bass model is widely used in market penetration analysis of a new product or service. The markets or social systems constitute a basic framework within which the adoption process of a technological innovation spreads. The appropriateness of applying a Bass concept to the extraction of oil resources relies at least on the strong positive correlation with the diffusion of the corresponding transportation technologies. The physicist Marchetti (1980) was one, among others, to implement the innovation diffusion paradigm, through logistic models, in the analysis of new energy resources.

The BM has been fully confirmed in quantitative marketing with reference to one life cycle products and was extended in Bass et al. (1994) with the introduction of a perturbation described by an integrable function $x(t)$ representing testable economic and strategic interventions.

The Generalised Bass Model, GBM, is then characterised by a non-autonomous Riccati equation, for $0 \leq t < +\infty$,

$$ {}_1z'(t) = \left(p + q\frac{{}_1z(t)}{m}\right)(m - {}_1z(t))x(t), \tag{4} $$

and its closed form solution, under initial condition ${}_1z(0) = 0$, is

$$ {}_1z(t) = m\frac{1 - e^{-(p+q)\int_0^t x(\tau)d\tau}}{1 + \frac{q}{p}e^{-(p+q)\int_0^t x(\tau)d\tau}} = m\,{}_1F(t), \qquad 0 \leq t < +\infty. \tag{5} $$

Function ${}_1F(t)$ is a perturbed distribution. Notice that market potential m, URR in an oil extraction context, will not affect peaking times and duration. It is a scale parameter. Function $x(t)$ modifies the geometry of adoptions over time and not the carrying capacity, m, or the diffusion parameters p and q. This aspect is not always preserved under inference, since asymmetric time evolution in extraction does not give rise to an 'optimal design' with known biases if the half-life span is not reached. We can model and test $x(t)$ in Equation (4) through some interventions, assuming a non-uniform distribution of their effects over time. Exponential types are usually effective, $x(t) = 1 + \sum_{i=1}^{3} c_i e^{b_i(t-a_i)} I_{t\geq a_i}$, in the case of memory effects due to external causes, e.g., permanent accumulating learning effects (positive b_i values) or, as an opposite situation, changes due to decaying effects (negative b_i values). For $x(t) = 1$, $0 \leq t < +\infty$, we obtain the standard Bass model, BM.

The GBM in Equation (5) with three exponential interventions was examined in Guseo et al. (2007) with reference to world oil production covering a century, 1900–2002 (Campbell's and BP's mean *daily* production data per year). If we extend the previous application, including annual data until 2008 from BP, we observe (see Table 1 comparing model *OneWave02* vs *OneWave08*) a relevant departure in URR estimation, $4.17e6$ vs. $5.28e6$, and a stable behaviour of the first part of the model, during the 1900–1980s, involving the estimation of well-known historical shocks (1951, 1973, and 1979).

The above-mentioned deviation in URR estimation may be imputed to a mixed production profile in the last two decades, suggesting a new approach based on a two-regime model. Two subsequent waves may be associated with two different oil 'densities' that we simply denote by 'cheap' and 'heavy' oils. Unfortunately, these two products are not systematically separated into different series. BP, for instance, considers a simple measure of aggregate oil production in barrels over time. We model previous mixed compounds with an aggregate two-regime model, i.e.,

$$ {}_2z(t) = m_c\,{}_1F(t) + m_h\,{}_2F(t - ch)I_{t\geq ch}, \tag{6} $$

where m_c is the URR of 'cheap' oil (essentially crude oil and a part of NGL), ${}_1F(t)$ depicts the dynamics of the first wave as in Equation (5), with three

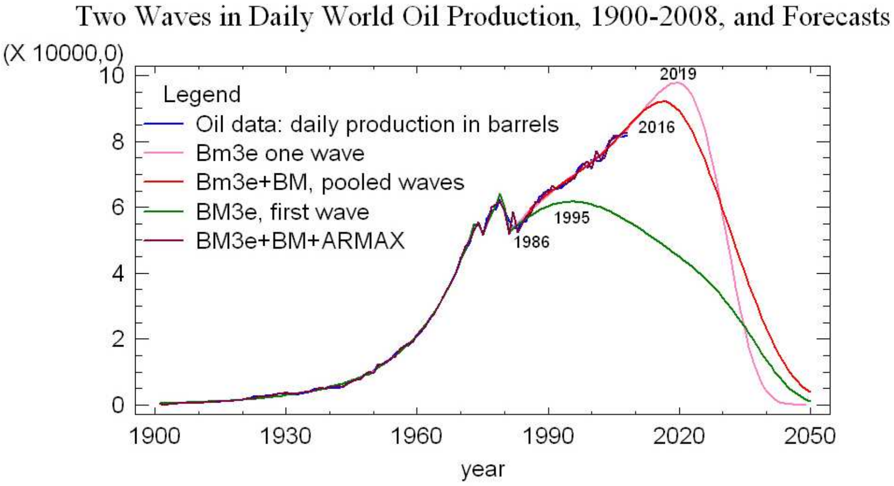

Figure 1: Comparison between a two-wave world oil model and the corresponding one-wave GBM. Relevant results for the two-wave model: 1986 is the estimated change of regime production, i.e., the origin of 'heavy' oil extraction; 1995 denotes the conventional peak oil production time; 2016 represents the new peaking time for mixed production. Comparatively, a one-wave model is worse and predicts a mixed behaviour in 2019. Data sources: Campbell and BP's daily production in thousands of barrels: 1900–2008.

exponential shocks, m_h is the URR of 'heavy' oils, and ${}_2F(t-ch)$ is a correspondingly *shifted* BM with parametric origin, ch. $I_{t\geq ch}$ is an indicator function (step function). Under this new approach, we expect a stable identification of well-known historical shocks which precede the more recent regime's change due to the mixed accounting behaviour.

Nonlinear least squares (NLS) results are reported in Table 1 for all the involved parameters, avoiding subjective external estimates of local URR and change point ch. If we examine the results of the proposed model, $TwoWave08$, we notice that $m \simeq m_c + m_h = 5.46e6$, or 1991 Gbo (Giga barrels of oil, if we refer to yearly production), is the estimated total URR related to these two cycles. The three historical shocks are perfectly recognised as such and a new interpretation of the current dynamics is plausible. We observe that 1986 is the estimated change point depicting the origin of 'heavy' oils extraction. The year 1995 represents the pure cheap oil peak,

Table 1: Results from GBM models with three shocks: TwoWave08 is a GBM with three shocks and a subsequent new heavy generation, BM, originating in 1986. First wave ('cheap' oil): $mc =$ URR; pc, $qc =$ innovative and imitative components; a_i, b_i, c_i, $i = 1, 2, 3$ are shock parameters. Second wave ('heavy' oil): originating in 1986 (ch); $mh =$ URR; ph, $qh =$ innovative and imitative components. For a comparison, consider two one-wave models: OneWave02, 1900–2002, Guseo et al. (2007); OneWave08, 1900–2008.

	mc	pc	qc	$c1$	$b1$	$a1$
TwoWave08	4.120330e+06	1.056836e-04	6.352327e-02	-3.004472e-01	6.344011e-02	8.056811e+01
OneWave08	5.280893e+06	8.249186e-05	6.334572e-02	-2.820173e-01	5.829027e-02	8.044347e+01
OneWave02	4.174551e+06	1.043901e-04	6.349702e-02	-3.218601e-01	5.674002e-02	8.050002e+01
	$c2$	$b2$	$a2$	$c3$	$b3$	$a3$
TwoWave08	7.229269e-02	7.174975e-02	5.105024e+01	-2.241371e-01	7.351826e-02	7.462345e+01
OneWave08	6.488241e-02	6.977838e-02	5.124170e+01	-2.281890e-01	6.479371e-02	7.458933e+01
OneWave02	7.177531e-02	7.187002e-02	5.107011e+01	-2.272032e-01	7.098011e-02	7.460001e+01
	mh	ph	qh	ch	SSE	R^2
TwoWave08	1.333679e+06	1.661451e-03	1.302502e-01	8.609658e+01	305920132	0.9999966
OneWave08					339274631	0.9999962
OneWave02					316947001	0.9999947

while 2016 denotes the new peaking time for aggregate production of 'cheap' and 'heavy' oils with about 93 md/b. Under a forced one-wave hypothesis, following Equation (5) with three shocks (see model *OneWave08* reported in Table 1), we obtain a weaker fitting, and the peak is located in 2019, with the steepest behaviour of the right tail. Figure 1 allows suitable comparisons among models, their components, and ARMAX sharpening of residuals. ARMAX sharpening is obtained under a hierarchical hypothesis on the relevance of different effects. Stochastic autocorrelated residuals $\varepsilon(t)$ have a limited impact, so that we solve a complex nonlinear least squares problem in the presence of autocorrelated errors with a two-stage procedure relying on the non-parametric robustness of NLS methodology determining the predicted values, ${}_2\hat{z}(t)$. If we consider the process $w(t) = {}_2z(t) + \varepsilon(t)$ where $\varepsilon(t)$ is an ARMA(p,q), then we can model $\Phi(B)(w(t) - d\,{}_2\hat{z}(t)) = \Theta(B)a(t)$, where $a(t)$ is a white noise, as a second stage estimate which allows a useful control on parameter d concerning the stability of first stage identification. An alternative simpler way assumes $d = 1$.

The proposed two-wave model results may be compared with the relevant literature. In particular, for an assessment of forecast date of peak, it may be informative that the estimates reported in Figure 3.7 in the technical report n.7 related to UKERC (2009). Our result, year 2016, defines a median position between pessimistic and optimistic approaches. Moreover, in a well-known report by Deutsche Bank, edited by Sankey et al. (2009), an equivalent result is obtained through different methodologies. In particular, in 2016 oil demand peaks with a price spike around US$ 175/bbl.

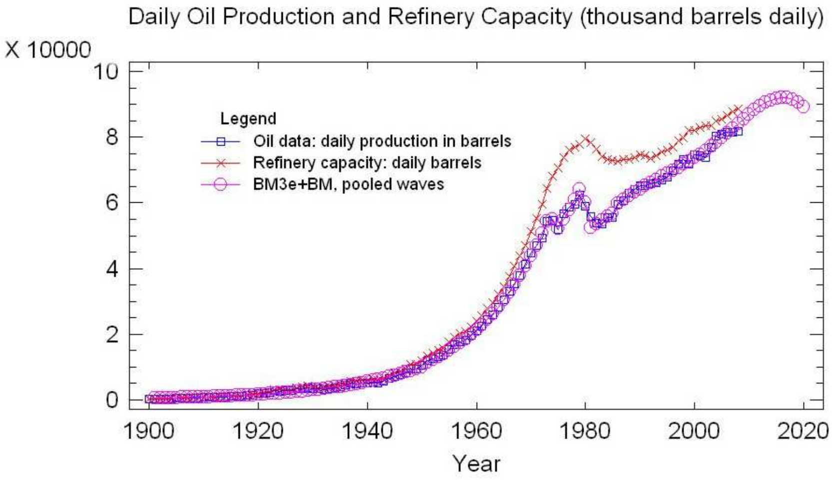

Figure 2: Comparison between world oil refining capacity, world oil production, and two-wave modelling. Data sources: Campbell and BP's daily production and capacity in thousands of barrels: 1900–2008.

As a final remark, we underline the theoretical and empirical necessity for introducing a two-wave model in order to take into account different dynamics of heterogeneous oil resources even if we deal with mixed production data.

Under previous results that emphasise an imminent mixed oil peak (2016) driven by heavier resources, it may be crucial to understand the dynamic behaviour of the refining processes as an essential tie among production, consumption, and environmental and strategic constraints.

3. Refining capacity: An emerging crunch?

High gasoline prices and related high diesel and heating oil prices are impacting the economy and consumers. They are likely driven by concerns about the availability of spare crude oil production capacity. A study of dynamics in refinery capacity may suggest a different interpretation. See, for example, ICF (2005).

Refinery 'spare capacity' has eroded in the last two decades, in the sense that the ratio between daily refinery capacity and daily oil production capacity is quite similar to the one that occurred in the decades preceding the well-known shocks in the 1970s (see, in particular, Figure 2).

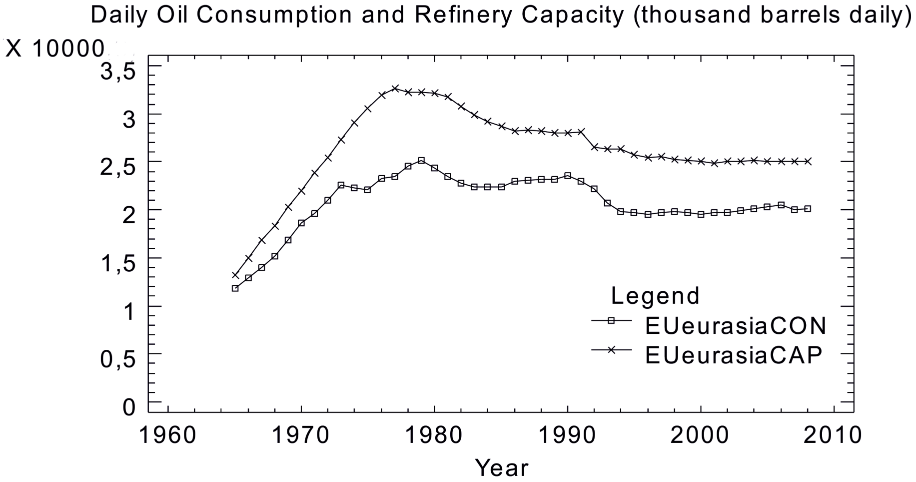

Figure 3: Comparison between EU-Eurasia oil refining capacity and corresponding consumption. Data source: BP's daily consumption and capacity in thousands of barrels: 1965–2008.

The increasing global demand for gasoline and diesel, and regulatory policies that impose lower sulfur content, are generating a trade-off between the demand for clean products and the availability of existing refining capacity from available heavier crude oil. This possible 'capacity crunch' is not only a theoretical guess. We observe a global chilling in refining capacity growth (see Figure 2), but there is some different behaviour at the regional level. For instance, in the U.S.A. the number of refineries fell to about 150 in 2009 from more than 300 in 1982 (see *The New York Times*, 24 December 2009).

Environmental concerns and corresponding legislation are blocking new refineries and major expansions in the OECD countries. In Europe we only have diesel hydrocracker additions to existing plants. For a joint assessment of oil consumption and oil refining capacity in Europe and Eurasia, see Figure 3.

On the other hand, China alone is growing its refinery capacity at a very high rate, with plants more oriented to heavy and sour oils (see Figure 4).

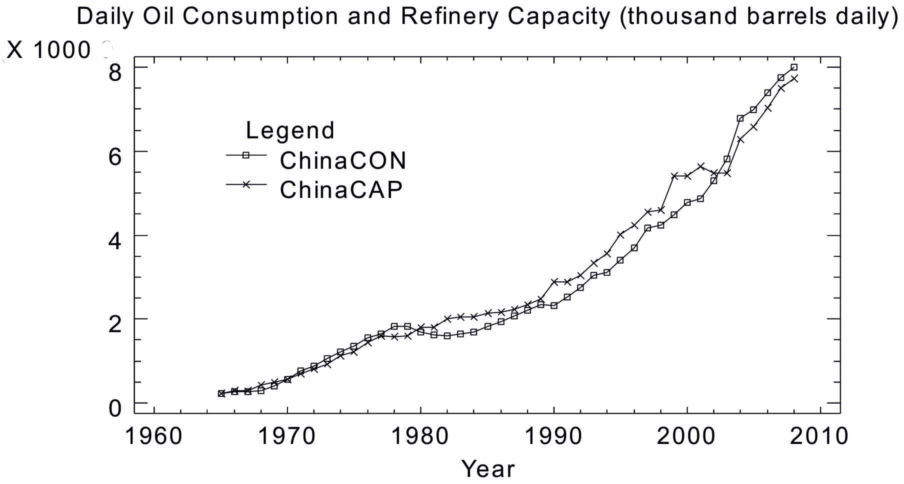

Figure 4: Comparison between China's oil refining capacity and corresponding consumption. Data source: BP's daily consumption and capacity in thousands of barrels: 1965–2008.

This strategy will increase competition, putting upward pressure on refining margins. This aspect may partially explain the observed reduced ratio between refining capacity, which implies long-term investments, and oil production, anticipating the risk of an imminent peak in oil production.

4. Conclusions

The proposed two-wave model for global oil production and forecasting considers the issue of evolutionary behaviour identification, taking into account aggregate time series. The main focus is on a sequential decomposition of observed aggregate production into two separate oil streams: conventional or 'cheap', light, sweet, regular oils, and unconventional or 'heavy' and sour oils.

The obtained results depend upon the assumed-observed behaviour and do not methodologically exclude the possible emergence of an 'extra-heavy' cycle with reference to a systematic development of Canadian oil sands, Venezuelan extra-heavy oils and, indirectly, in the exploitation of shale gas with increasing environmental costs (see, for example, Walsh 2011).

The parallel assessments of refinery capacities at global and regional levels emphasise the observed investment contraction, especially with reference to the conventional oils, and a limited expansion of heavy and sour oil plants in emerging countries.

Appendix A.

With simple algebra it is easy to prove, for $-\infty < t < +\infty$, that the normalised version of logistic density by Maggio and Cacciola (2009) is

$$\begin{aligned} {}_kg(t) &= \frac{rf(k)e^{-r(t-t_p)}}{2(1-k)e^{-r(t-t_p)} + k\left(1+e^{-r(t-t_p)}\right)^2} \\ &= \frac{\frac{r}{2}f(k)}{(1+k\cosh(r(t-t_p)))}, \qquad 0 < k < 1, \end{aligned} \tag{A.1}$$

where $f(k) = \sqrt{(}1-k^2)/\ln(k/(1-\sqrt{(}1-k^2)))$ is a normalising factor. For $k \to 1$, the function $f(k)$ converges to 1, $\lim_{k\to 1} f(k) = 1$ and, therefore, Equation (A.1) converges to the standard logistic density.

The corresponding distribution function is,

$${}_kG(t) = \int_{-\infty}^{t} {}_kg(x)dx = \frac{\ln\left(\frac{(\sqrt{1-k^2}+1)^2(ke^{rt}+e^{rt_p}(1-\sqrt{1-k^2}))}{k^2(ke^{rt}+e^{rt_p}(\sqrt{1-k^2})+1))}\right)}{\ln\left(\frac{(\sqrt{1-k^2}+1)^2}{k^2}\right)} \tag{A.2}$$

where, in particular, $\lim_{t\to+\infty} {}_kG(t) = 1$.

The differential equation that characterises ${}_kg(t)$ as a function of ${}_kG(t)$ is,

$$\begin{aligned} {}_kg(t) &= \frac{rf(k)E}{2(1-k)E + k\left(1+E\right)^2}, \qquad \text{(A.3)} \\ E &= \frac{1-C}{C\left(\frac{1-\sqrt{1-k^2}}{k} - \frac{\sqrt{1-k^2}+1}{kC}\right)}, \\ C &= \left(\frac{(\sqrt{1-k^2}+1)^2}{k^2}\right)^{1-{}_kG(t)}. \end{aligned}$$

In order to check whether Equation (A.2) is a proper solution to Equation (A.3), we substitute Equation (A.2) in C definition obtaining

$$C = \left(\frac{(\sqrt{1-k^2}+1)^2}{k^2}\right)^{1-{}_kG(t)} = \frac{ke^{rt}+e^{rt_p}(\sqrt{1-k^2})+1)}{ke^{rt}+e^{rt_p}(1-\sqrt{1-k^2})}, \tag{A.4}$$

and, therefore,

$$E = \frac{1-C}{C\left(\frac{1-\sqrt{1-k^2}}{k} - \frac{\sqrt{1-k^2}+1}{kC}\right)} = e^{-r(t-t_p)}, \tag{A.5}$$

that verifies Equation (A.1).

Equation (A.3) establishes a nonlinear relationship between normalised cumulative function ${}_kG(t)$ and the corresponding rate or density function ${}_kg(t)$. Its interpretation from a substantive point of view is not simple as in the logistic limiting case for $k \to 1$.

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