A Class of Automata Networks for Diffusion of Innovations Driven by Riccati Equations

Renato Guseo[1] Mariangela Guidolin[2]

[1]Department of Statistical Sciences
University of Padova, Italy

[2]Department of Economics
University of Padova, Italy

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Outline

Abstract

Automata Networks for Diffusion of Innovations 1

- Innovation diffusion processes are generally described at aggregate level with models like the Bass model (1969) and the Generalized Bass Model (1994).
- However, the recognized importance of communication channels between agents has recently suggested the use of agent-based models, like Cellular Automata.
- We argue that an adoption process is nested in a communication network that evolves dynamically and implicitly generates a non–constant potential market.

Abstract

Automata Networks for Diffusion of Innovations 2

- Using Cellular Automata we propose a two–phase model. First, we describe the Communication Network necessary for the awareness of an innovation. Then, we model a nested process representing adoption dynamics.
- Through a "Mean Field Approximation" we propose a combined continuous representation of the discrete time equations derived by our Automata Network. This constitutes a special non autonomous Riccati equation. The main results refer to the closed form solution and to the corresponding statistical analysis for identification and inference.
- We discuss an application in the field of bank services.

Aggregate Differential Equations Approach 1

Diffusion of Innovations Approaches

- Identification and inference in existing life cycles.
- Riccati (1676-1754), Verhulst (1838)
- Fourt and Woodlock (1960), Mansfield (1961), Bass (1969)

$$z'(t) = \left(p + q\frac{z(t)}{m}\right)(m - z(t)). \tag{1}$$

Instantaneous adoptions, $z'(t)$; cumulative adoptions, $z(t)$; constant potential market, m; residual market $(m - z(t))$; coefficient of innovation, p; coefficient of imitation, q; penalizing relative knowledge, $z(t)/m$, (w.o.m. effect).

Aggregate Differential Equations Approach 2

Diffusion of Innovations Approaches

- Mahajan & Muller(1979), Mahajan, Muller & Bass (1990).
- The Generalized Bass Model (GBM) by Bass, Krishnan and Jain (1994) includes exogenous interventions (strategic decisions, policies, marketing strategies),

$$z'(t) = \left(p + q\frac{z(t)}{m}\right)(m - z(t))x(t), \qquad (2)$$

  where $x(t)$ denotes a general function with neutral level 1, whose effect acts on adoptions' timing but cannot control the constant potential m or the parameters p and q. Guseo, Dalla Valle and Guidolin (2007).
- Mahajan, Muller and Wind (2000), Rogers (2003), Meade and Islam (2006) Mahajan, Muller and Peres (2007).

Aggregate Differential Equations Approach 3

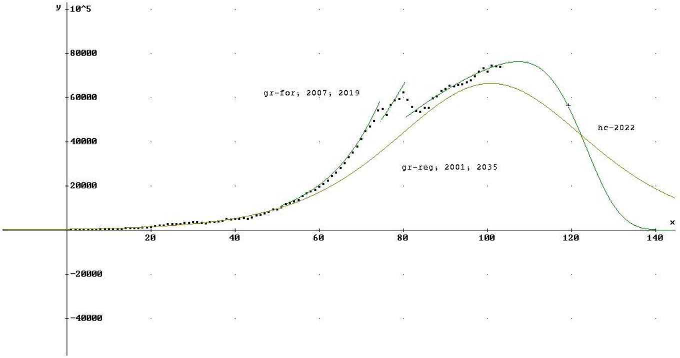

Figure: World oil depletion: GBM with three shocks vs. Hubbert–Bass model (ex Guseo, Dalla Valle and Guidolin, *Technological Forecasting and Social Change*, 74(4), 452–469).

Interlude: dynamic potential and absorptive capacity

Diffusion of Innovations Approaches

- Relationship between information and innovation diffusion;
- Qualitative approach by Cohen and Levinthal (1990);
- absorptive capacity: "ability to recognize the value of new information, assimilate it and apply it";
- Current absorptive capacity depends upon the accumulation of prior related knowledge;
- The development of a collective knowledge may be thought of as an evolving network.

Agent Based Models

Diffusion of Innovations Approaches

- Different communication channels, different radii
  - Cellular Automata, Network Automata, Wolfram (1983),
  - Simulations (Game of life: John H. Conway (1970))
  - Boccara (2004), Goldenberg and Efroni (2001)
  - Goldenberg, Libai and Muller (2001), Moldovan and Goldenberg (2004).
- Spatial correlation
  - local interaction (micro), bottom–up self organization, local to global mapping (macro): emerging global behaviour
  - inverse problem: local rule from a given global behaviour
- Relationships between two approaches
  - learning from evolution , Guseo and Guidolin (2007)
  - mean field approximation vs differential representation.

Cellular Automata: some definitions

- A Cellular Automata model (CA) is composed of a grid of cells, $i \in Z$ (the set of all integers) and each one of them is in a specific state (e.g. adopter=1, neutral=0). According to Boccara and Fukś (1997) and (1999) define $s(i,t)$ the state of the cell i at time t.
- The change of state is governed by a transition rule (deterministic or stochastic) which synthesizes the local interactions of ray v between a cell and its range of interaction:
  $s(i,t+1) = f(s(i-v,t), s(i-v+1,t), \cdots, s(i,t), s(i+1,t), \cdots, s(i+v,t))$.
- A wider class of automata, Network Automata (NA) considers function f as i–dependent and with asymmetrical and variable neighborhood.

A Communication Network 1

- Let $G = (V, E)$ be a *finite* directed graph, where $V = \{1, 2, \cdots, i, \cdots, N\}$ is a set of vertices whose cardinality is $N = \mathbf{c}(V)$.
- The set E of ordered pairs (i, j) called *directed edges* or arcs, $E \subset V \times V$, depicts a subset of all possible binary relationships within vertices V including reflexive relationships. Cardinality of E is $U = \mathbf{c}(E) \leq N^2$.
- In social or physical systems these constraints are based on large distances or accessibility limits.
- We denote the *state* of an edge (i, j) at time t with an indicator function $c(i, j; t)$. Function $c(i, j; t)$ equals 1 if and only if the edge (i, j) is active, i.e., when information about an innovation is transmitted, otherwise is zero, in particular, if $(i, j) \notin E$.

A Communication Network 2

- $$\sigma_c(i,j;t) = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} c(i+n, j+m; t) p_{n,m}\,; \qquad \sum_{n,m} p_{n,m} = 1. \tag{3}$$

  where $p(n, m)$ is a probability distribution. Note that $\sigma_c(i, j; t)$ itself is a probability.
- If we assume that this local pressure is *translational invariant*, we may consider the "Mean Field Approximation" that excludes the local effect of distribution $p_{n,m}$,

$$\sigma_c(i,j;t) \simeq \nu(t) = \sum_{i,j} \frac{c(i,j;t)}{U}. \tag{4}$$

A Stochastic Transition Rule in the Network

- $$\begin{aligned} c(i,j;t+1) &= c(i,j;t) + Bi(1,p_c)\, I_{(c(i,j;t)=0)} + \\ &+ Bi(1, q_c\, \sigma_c(i,j;t))\, I_{(c(i,j;t)=0)} + \\ &- Bi(1,e_c) I_{(c(i,j;t)=1)} - Bi(1, w_c \sigma_c(i,j;t))\, I_{(c(i,j;t)=1)} \end{aligned} \quad (5)$$
- $Bi(1, p_c)$, $Bi(1, q_c\, \sigma_c(i,j;t))$, $Bi(1, e_c)$, $Bi(1, w_c\, \sigma_c(i,j;t))$ are binomial experiments,
- Parameter p_c and q_c denote usual Bass input effects in network expansion. Analogously, e_c and w_c depict decay effects in the network.

Network Automata and Mean Field Approximation

- Let us consider, therefore, the average number of active edges within E at time t following the mean behaviour of the transition rule (5),

$$\begin{aligned} U\nu(t+1) &= U[\nu(t)+p_c(1-\nu(t)) \\ &+ q_c\nu(t)(1-\nu(t))-e_c\nu(t)-w_c\nu^2(t)]. \end{aligned} \quad (6)$$

- We can approximate previous discrete time equation with a continuous Riccati equation, namely,

$$\nu'(t) = -(q_c+w_c)\nu^2(t)+(q_c-p_c-e_c)\nu(t)+p_c. \quad (7)$$

- Solution $\nu(t)$ of previous Equation (7) is described in Appendix A as a special case for $f(\cdot)=g(\cdot)=1$.

Potential Market Definition

- Function $U\nu(t)$ defines an aggregate temporal knowledge evolution of an innovation based on active edges. This is only a preliminary step in absorptive capacity definition.
- The positive squared root of $U\nu(t)$,

$$k(t) = \sqrt{U}\sqrt{\nu(t)}, \tag{8}$$

depicts the upper bound of the *carrying capacity* $m(t)$ for the related adoption process by individuals describing the system, the vertices of the graph $G = (V, E)$

- In order to describe possible replications in adoption we consider a more flexible parameter K

$$m(t) = K\sqrt{\nu(t)} \tag{9}$$

A Cellular Automaton for Adoption Process

- We denote the state of a vertex $i \in V$ at time t with the indicator function $s(i; t)$. We define a stochastic transition rule for the description of an individual adoption process over time with the notation of cellular automata, i.e.,
-

$$
\begin{aligned}
s(i; t+1) &= s(i; t) + Bi(1, p_s)\, I_{(s(i;t)=0)} + \\
&+ Bi(1, q_s\, \sigma_s(i; t))\, I_{(s(i;t)=0)} + \\
&- Bi(1, r_s)\, I_{(s(i;t)=1)} + \\
&+ s(i; t) . \frac{m'(t)}{m(t)} .
\end{aligned}
\tag{10}
$$

- Experiments: $Bi(1, p_s)$, $Bi(1, q_s\, \sigma_s(i; t))$, $Bi(1, r_s)$
- Parameters denote adoption, p_c and q_c, or dis–adoption effects, r_s.

A Mean Field Approximation of CA

- The average behaviour of Equation (10) followed by a summation over all the states $s(i;t)$ within V is a discrete time *co–evolutive model*

$$\begin{aligned} y(t+1) &= y(t) + p_s(m(t) - y(t)) + q_s \frac{y(t)}{m(t)}(m(t) - y(t)) \\ &- r_s y(t) + y(t)\frac{m'(t)}{m(t)}. \end{aligned} \tag{11}$$

- A continuous approximation of previous Equation (11) is

$$\begin{aligned} y'(t) &= m(t)\left\{-r_s \frac{y(t)}{m(t)} + \left(p_s + q_s \frac{y(t)}{m(t)}\right)\left(1 - \frac{y(t)}{m(t)}\right)\right\} \\ &+ y(t)\frac{m'(t)}{m(t)}. \end{aligned} \tag{12}$$

A Perturbed Evolution of an Adoption Process

- We model this more flexible context multiplying by an impact function, $x(t)$, whose neutral level is obviously $x(t) = 1 \, \forall t$, i.e.,

$$\begin{aligned} y'(t) &= m(t)\left\{-r_s\frac{y(t)}{m(t)} + \left(p_s + q_s\frac{y(t)}{m(t)}\right)\left(1 - \frac{y(t)}{m(t)}\right)\right\}x(t) \\ &+ y(t)\frac{m'(t)}{m(t)}. \end{aligned} \tag{13}$$

- This is a special Riccati equation (Appendix A). Note that $x(t)$ exerts its effect only on the first component of Equation (13) which is a function of the residual market.

Statistical Co–evolutive Modelling and Interventions 1

Dynamic Potential Market

- In this sense we have to determine, preliminarily, the potential market $m(t)$ on the basis of Equation (7) and Equation (16)(see Appendix A). For the initial conditions $m(0) = 0$, $f(\cdot) = 1$ and $g(\cdot) = 1$, we obtain

$$m(t) = K\sqrt{\frac{1 - e^{-D_c t}}{\frac{1}{{}_c r_2} - \frac{1}{{}_c r_1} e^{-D_c t}}}, \qquad (14)$$

- where $D_c = \sqrt{(q_c - p_c - e_c)^2 + 4(q_c + w_c)p_c} > 0$, ${}_c r_i = (-(q_c - p_c - e_c) \pm D_c)/(-2(q_c + w_c)),\; i = 1, 2$, with ${}_c r_2 > {}_c r_1$. If, for instance, $e_c > 0$ then the limit of $m(t)$ for $t \to +\infty$ may be less than K.

Statistical Co–evolutive Modelling and Interventions 2

Co–evolutive Adoption Process

- Under an initial condition $C = 0$, for $g(\cdot) = m(\cdot)$ and $f(\cdot) = x(\cdot)$ the perturbed co–evolutive model, controlled by Equation (13) is determined on the basis of Equation (16) (see Appendix A),

$$y(t) = m(t) \frac{1 - e^{-D_s \int_0^t x(\tau) d\tau}}{\frac{1}{{}_s r_2} - \frac{1}{{}_s r_1} e^{-D_s \int_0^t x(\tau) d\tau}}, \tag{15}$$

- where $D_s = \sqrt{(q_s - p_s - r_s)^2 + 4 q_s p_s} > 0$ and ${}_s r_i = (-(q_s - p_s - r_s) \pm D_s)/(-2q_s),\ i = 1, 2$, with ${}_s r_2 > {}_s r_1$.

Statistical Co–evolutive Modelling and Interventions 3

Statistical Aspects

Implementation of model (15) may adopt different error structures:

- a nonlinear regressive approach, $w(t) = y(t) + \varepsilon(t)$, with an i.i.d. residual $\varepsilon(t)$.
- a complementary approach is based on ARMAX representation with a standard nonlinear estimation as a first step.
- Joint identifiability of parameters in Equation (7) is not possible, so that we have to evaluate which are the dominant effects. A common choice is e_c or w_c exclusion.

Co-evolutive Plot

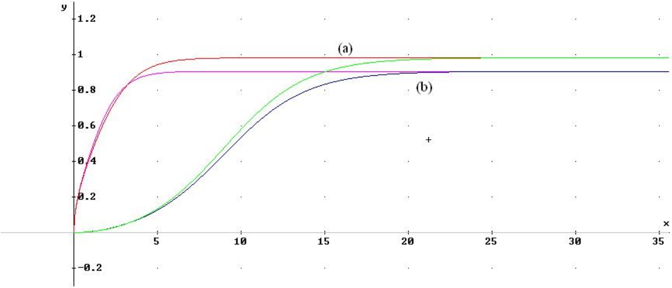

Figure: Two different communication frameworks. Common adoption parameters: $q_s = 0.4,\ p_s = 0.01,\ r_s = 0$; Common communication parameters: $K = 1,\ p_c = 0.15,\ e_c = 0.03$. Special cases: Case $(a) : q_c = 0.7,\ w_c = 0$, Case $(b) : q_c = 0.9,\ w_c = 0.2$.

Current Account Diffusion 1

- We examine the weekly cumulative diffusion of a particular bank current account introduced by Cardine in a northern area of Italy (Area 2) for small and medium size firms.
- The cumulative data refer to a 64 weeks period from the origin of the service.
- Original data inspection suggests us that the exit rules parameters at both levels (communication network and adoption process) may be considered, at a first step, non significant, i.e., $e_c = w_c = r_s = 0$.
- Following these assumptions we implement our model (15) with variable potential in order to understand its performance under a nonlinear regressive framework. The main results are outlined in Table 1.

Current Account Diffusion 2

Table: Current account diffusion. Parameters estimates of co–evolutive model for Cardine Area 2 data with no exit rule. ( ) marginal linearized asymptotic 95% confidence limits

K	q_c	p_c	q_s	p_s	R_1^2	$D-W$
6883.1	0.1840	0.1730	-0.0164	0.0192	0.998265	0.444
(-9508)	(-0.357)	(-0.030)	(-0.0721)	(-0.0252)	SSE :	
(23274)	(0.725)	(0.376)	(0.0394)	(0.0636)	[135785]	

Current Account Diffusion 3

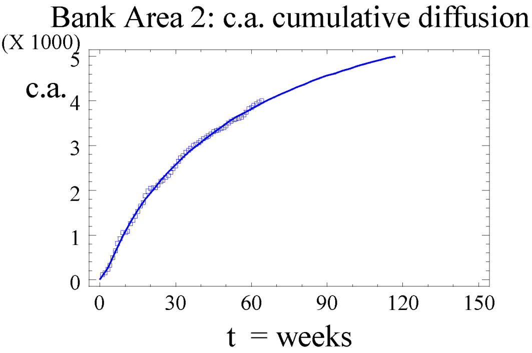

Figure: Current account diffusion (Area 2, Cardine, Italy). Co–evolutive cumulative model with no exit rule.

Current Account Diffusion 4

- Under such conditions the marginal linearized asymptotic 95% confidence intervals are instable so that we may exclude their marginal direct use.
- Nevertheless, global use of transfer function is unaffected.
- We argue that this problem may be overcome by implementing an appropriate ARMAX procedure. The main results are outlined in Table 2.

Current Account Diffusion 5

Table: Co-evolutive cumulative model with no exit rule and ARMAX(2,0,0) sharpening. ( ) *t*–statistic; [ ] *p*–values

$AR(1)$	$AR(2)$	$PREb2cobs000$	$mean$	SSE
1.0729	-0.3886	0.3080	118.538	45616
(9.422)	(-4.569)	(5.4987)	(2.8222)	$\{d.f.61\}$
[0.000000]	[0.000025]	[0.000001]	[0.006430]	$R_2^2 = 0.999427$

Current Account Diffusion 6

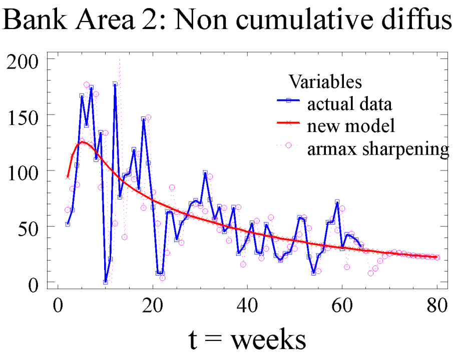

Figure: Current account diffusion (Area 2, Cardine, Italy). Co–evolutive non cumulative model with no exit rule, ARMAX sharpening and actual "active bank account" data.

Current Account Diffusion 7

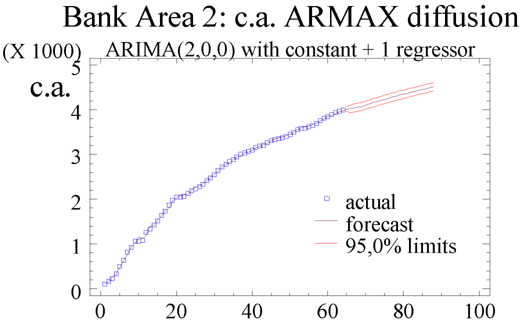

Figure: Current account diffusion (Area 2, Cardine, Italy). Co–evolutive cumulative model with no exit rule and cumulative ARMAX sharpening.

Current Account Diffusion 8

- Following these results, evaluation of dynamic potential market – which is essentially a latent structure that we can not measure directly – may be compared with the approximate averaged dynamics described by model (15).
- As we can see, by inspecting Figure 6, the inferred potential market reaches its stationary level after ten weeks demonstrating that the joint communication and marketing effort effects are very rapid.
- Both parameters p_c and q_c are quite high: 0.173 and, respectively, 0.184, with an expected high value for p_c that represents the direct bank communication effort.

Current Account Diffusion 9

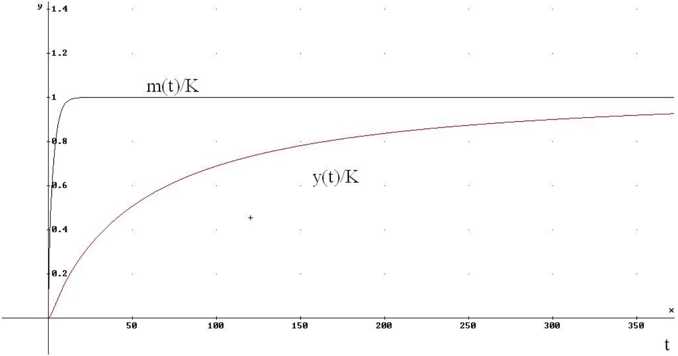

Figure: Normalized current account diffusion (Area 2, Cardine, Italy). Co–evolutive cumulative model with no exit rule and normalized dynamic potential market.

Final Remarks and Discussion 1

a) Innovation diffusion is not a univariate adoption process over time. We argue that an adoption process is nested in a communication network that evolves dynamically generating the corresponding non–constant potential market.

b) We guess that a communication network is a necessary phase in determining the evolution of a *prior related knowledge*, which is, using Cohen and Levinthal's (1990) terminology, the basic element for developing an *absorptive capacity*.

c) Our two-phase modelling is a mathematical and statistical specification of the above general ideas in order to test performances, significance of model components and forecasting.

Final Remarks and Discussion 2

d) Cellular Automata and Network Automata are a simple and effective tool for representing both the communication network evolution and the nested adoption process.

e) In our model we assume that the communication network is not observable. In general we do not have precise information about how agents communicate between them and the network we consider has a virtual structure. However we are not interested in determining detailed particular shapes of the actual network. Our focus is on an aggregate transformation of this network, i.e., the concrete potential market $m(t)$.

f) With a *mean field approximation* we have reformulated a Complex Systems representation in a dual tractable differential one, see Equations (7), (9) and (13).

Final Remarks and Discussion 3

g) In our model, especially with reference to the Riccati Equation (13), functions $m(t)$, and $x(t)$ are independent tools. The effect of intervention function $x(t)$ modifies the time path of diffusion by locally expanding or shrinking adoptions within a "balance equation constraint". Instead, the potential market, $m(t)$, controls and modifies the size of this process, expressed in terms of the absolute amount of adoptions. This is a technical specification, useful to avoid theoretical misunderstandings between these different and separable effects.

Final Remarks and Discussion 4

h) The proposed application gives some insights on the role of statistics in analyzing evolving time series within a life cycle context. In particular, we observe that in this specific application the Mean Field Approximation, that allows an interesting aggregate description of a Complex Systems representation, does not consider effects of a supposed (not observed) heterogeneity of adopters (or adoptions). Nevertheless, an ARMAX sharpening, applied as a second step after a nonlinear least squares procedure, completes inference in a satisfactory way.

Final Remarks and Discussion 5

i) The substantive implication of our model, is that we are able to estimate, in an indirect way and under appropriate theoretical assumptions, the character of an evolving potential market simply using cumulative selling data. This is of particular concrete interest because it allows to measure indirectly the receptiveness of a social context, facilitating comparisons between different situations and evaluations on the effectiveness of firms' marketing efforts.

Final Remarks and Discussion 6

- Existence of a clear link between special classes of Network Automata – Cellular Automata and Riccati Equation including Bass family (BM, GBM);
- an answer to the inverse problem (Ganguly et al (2003)): "find a Cellular Automata rule that will have some preselected global properties";
- we avoid Genetic Algorithms which may reach heavy computational complexities without a simple interpretable theoretical framework (see, Venkatesan et al. (2004)).
- Outlook
  - alternative descriptions of dynamic potential markets: chilling effects, network externalities, etc. ;
  - alternative inferential techniques: joint NLS-ARMAX estimation, maximum likelihood, stochastic interventions.

Market receptiveness as a function of knowledge

- Ability to understand and recognise the value of a new product: the role of innovators, the role of word-of-mouth effect.
- Ability to discover and exploit the existence of a communication network: spread the message! The role of the firm in information choices.
- Different strategies: two similar products, the same market.
- Different communication structures and levels of receptiveness: two (or more) markets, the same product.
- Different stages of adoption: successive generations of the same product in one market. How the communication structure has evolved: acceleration of communication processes, shortened life cycles.

APPENDIX A: A Riccati Equation

- Let us consider the following special Riccati equation in (X,Y) real space

$$y'_x = a\frac{f(x)}{g(x)}y^2 + \left(bf(x) + \frac{g'(x)}{g(x)}\right)y + cf(x)g(x), \qquad (16)$$

  where $a, b, c \in R$, $D = \sqrt{b^2 - 4ac} > 0$,
  $r_i = (-b \pm D)/2a \in R, i = 1, 2$ and $g(x) \neq 0$, $f(x)$ are real functions.

- If the initial condition is set to zero, $C = 0$, we obtain,

$$y(x) = g(x)\frac{1 - e^{-D\int_0^x f(\tau)d\tau}}{\frac{1}{r_2} - \frac{1}{r_1}e^{-D\int_0^x f(\tau)d\tau}}. \qquad (17)$$

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