Cellular Automata and Riccati Equation Models for Diffusion of Innovations

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Torino, 14-16 June 2006/ XLIII Riunione Scientifica, Società Italiana di Statistica

Outline

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1. Diffusion of Innovations
   - Aggregate Differential Approach
   - Agent Based Models
2. Cellular Automata and Mean Field Approximation
3. A Riccati Equation
4. Interventions, Statistical Modeling and Applications
   - Model building and Inference
   - Bank Account Diffusion
   - Italian Tricars Evolution

Aggregate Differential Approach

Diffusion of Innovations

- Identification and inference in existing life cycles.
- Verhulst (1838)
- Bass (1969)
- Mahajan, Muller and Bass (1990)
- Bass (1994)
- Rogers (1995)
- Meade and Islam (2006)

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Agent Based Models

Diffusion of Innovations

- Different communication channels, different radii
  - Cellular Automata, Network Automata
  - Simulations (Game of life: John H. Conway (1970))
  - Wolfram (1983)
  - Goldenberg, Libai and Muller (2001)
- Spatial correlation
  - local interaction (micro), bottom–up self organization, local to global mapping (macro): emerging global behaviour
  - inverse problem: local rule deduction from a given global behaviour
- Relationships between two approaches
  - learning from evolution
  - mean field approximation vs differential representation.

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Cellular Automata

- A Cellular Automata model (CA) is composed of a grid of cells, $i \in Z$ (the set of all integers) and each one of them is in a specific state (e.g. adopter=1, neutral=0). According to Boccara and Fukś (1999) define $s(i,t)$ the state of the cell i at time t.
- The change of state is governed by a transition rule (deterministic or stochastic) which synthesizes the local interactions of ray v between a cell and its range of interaction:
$s(i,t+1) = f(s(i-v,t), s(i-v+1,t), \cdots, s(i,t), s(i+1,t), \cdots, s(i+v,t))$.
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Cellular Automata

- A kind of neighboring *pressure*:

$$\sigma(i,t) = \sum_{n=-\infty}^{\infty} s(i+n,t)p(n), \qquad \sum_{n=-\infty}^{\infty} p(n) = 1, \quad p(n) \geq 0, \tag{1}$$

  where $p(n)$ is a probability distribution. Note that $\sigma(i,t)$ itself is a probability.

- A first tentative model describing f is defined via a transition probability matrix:

$$\begin{pmatrix} P_{0\leftarrow 0} & P_{0\leftarrow 1} \\ P_{1\leftarrow 0} & P_{1\leftarrow 1} \end{pmatrix} = \begin{pmatrix} 1-\sigma(i,t) & 0 \\ \sigma(i,t) & 1 \end{pmatrix} \tag{2}$$

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- Define $\rho(t)$ the average density of adopters at time t and $\eta(t) = 1 - \rho(t)$ the average density of neutrals, with spatial averaging $\langle\ \rangle$ on pressure $\sigma(i,t)$:

$$\begin{pmatrix} \eta(t+1) \\ \rho(t+1) \end{pmatrix} = \begin{pmatrix} \langle P_{0\leftarrow 0}\rangle & \langle P_{0\leftarrow 1}\rangle \\ \langle P_{1\leftarrow 0}\rangle & \langle P_{1\leftarrow 1}\rangle \end{pmatrix} \cdot \begin{pmatrix} \eta(t) \\ \rho(t) \end{pmatrix}, \tag{3}$$

and therefore

$$\rho(t+1) = \rho(t) + (1-\rho(t))\langle\sigma(i,t)\rangle. \tag{4}$$

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Cellular Automata and Mean Field Approximation

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- $$\begin{pmatrix} \eta(t+1) \\ \rho(t+1) \end{pmatrix} = \begin{pmatrix} 1-q\langle\sigma(i,t)\rangle & r \\ q\langle\sigma(i,t)\rangle & 1-r \end{pmatrix} \cdot \begin{pmatrix} \eta(t) \\ \rho(t) \end{pmatrix} \tag{6}$$
  and, therefore,
  $$\rho(t+1) = (1-r)\rho(t) + q\langle\sigma(i,t)\rangle(1-\rho(t)). \tag{7}$$
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Cellular Automata and Mean Field Approximation

- Recovering of initializing aspects of diffusion,

$$\langle P_{1 \leftarrow 0} \rangle = p + q \langle \sigma(i,t) \rangle, \tag{9}$$

  where p denotes the probability of an external information pressure due to media;

- Approximation of finite difference $\rho(t+1) - \rho(t)$ with the prime derivative, $\rho'(t)$. The proposed limiting continuous time BFG model under the Mean Field Approximation is

$$\rho'(t) = \rho(t+1) - \rho(t) = -r\rho(t) + (p + q\rho(t))(1 - \rho(t)). \tag{10}$$

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Separable non Autonomous Riccati Equation

$$y' + (ay^2 + by + c)x(t) = 0, \quad a, b, c \in R, \quad I(t) = \int_0^t x(\tau)d\tau < \infty, \tag{11}$$

where $x(t)$ is an integrable intervention function, $y(0) = 0$, $r_i = (-b \pm \sqrt{b^2 - 4ac})/2a \in R, i = 1, 2$, with $D = a(r_2 - r_1) = \sqrt{b^2 - 4ac} > 0$ and $r_2 > r_1$ so that an equivalent representation is $y' + a(y - r_1)(y - r_2)x(t) = 0$.

$$y(t) = \frac{1 - e^{-a(r_2 - r_1)\int_0^t x(\tau)d\tau}}{\frac{1}{r_2} - \frac{1}{r_1} e^{-a(r_2 - r_1)\int_0^t x(\tau)d\tau}}. \tag{12}$$

If $\lim_{t\to\infty} I(t) = +\infty$, we attain a limiting behaviour of $y(t)$, i.e., $\lim_{t\to\infty} y(t) = r_2$.

Outline

Model Building and Inference

$$y' = \{-ry+(p+qy)(1-y)\}x(t), \quad r, p, q, \in R, \ 0 < p < q, \ 0 < r. \tag{13}$$

- Extended Boccara-Fuks model, BFG, is a special case under $x(t) = 1$;
- Generalized Bass model, GBM, by Bass *et al.* (1994) is attained for $r = 0$;
- Standard Bass model, BM, by Bass (1969) includes both the constraints, $x(t) = 1$ and $r = 0$.

Solution

- The factorization of Equation (13) is

$$y' + q(y - r_1)(y - r_2)x(t) = 0, \tag{14}$$

with real roots, for $D = \sqrt{(r + p - q)^2 + 4pq} > 0$, equal to

$$r_i = \frac{-(r + p - q) \pm D}{2q}, \quad i = 1, 2, \tag{15}$$

- We have $a(r_2 - r1) = D$ and the closed form solution is

$$y(t) = \frac{1 - e^{-D\int_0^t x(\tau)d\tau}}{\frac{1}{r_2} - \frac{1}{r_1}e^{-D\int_0^t x(\tau)d\tau}}. \tag{16}$$

NLS-ARMAX

- Absolute scale representation of natural diffusion, $z(t) = My(t)$,
- Asymptotic behaviour:
  - $m = \lim_{t\to+\infty} z(t) = Mr_2$.
  - Asymptotic fraction of innovative adoptions $F_{p,r}(t \to +\infty) = \frac{p}{qr_2} \log\left(1 + \frac{q}{p}\right)$ for $r_2 > 0$.
- Stochastic components:
  - $w(t) = z(t) + \varepsilon(t)$, with an i.i.d. residual $\varepsilon(t)$.
  - For autocorrelated residual errors is based combined statistical technique, namely NLS–ARMAX (see e.g. Guseo (2004), Guseo and Dalla Valle (2005) and Guseo *et al.* (2006)).

Outline

Bank Account Diffusion

- Weekly stock diffusion, first 64 weeks, of a special bank current account introduced by Cardine in north–east regions of Italy for small or medium size firms;
- Model BFG defined by equation (10) and one exponential shock embedded in intervention function, $x(t) = 1 + c_1 e^{b_1(t-a_1)}$: Structural change in economic and administrative facilities which occurred in the first part of the cycle.

Bank Account Diffusion

Table: Cardine current account diffusion: north–east Italy. BFG vs GBM.

	M	r	m $[Mr_2]$	p	q	c_1	b_1	a_1	R^2
BFGe1	21872 (-163246) (206991)	0.0067 (-0.380) (0.394)	16577	0.0019 (-0.0138) (0.0176)	0.0259 (-0.2594) (0.3112)	3.67 (1.39) (5.96)	-0.187 (-0.326) (-0.048)	8.4 (7.5) (9.3)	0.99871
GBMe1			12536 (8667) (16405)	0.0036 (0.002) (0.004)	0.0234 (0.0163) (0,0306)	3.955 (2.58) (5.32)	-0.255 (-0.354) (-0.156)	9.2 (8.3) (10.1)	0.99858

( ) lower and upper 95% linearized asymptotic confidence limits.

Bank Account Diffusion

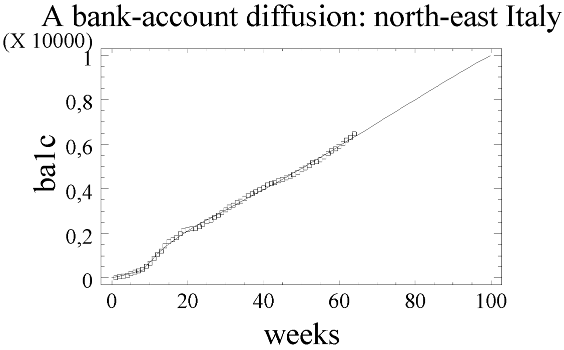

Figure: A current account diffusion: north–east Italy: BFG (zoom) ; Data source: Cardine.

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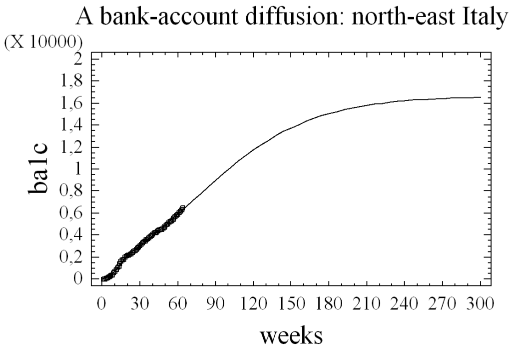

Figure: A current account diffusion: north–east Italy: BFG; Data source: Cardine.

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Italian Tricars Evolution

- Evolution of annual Italian tricars stock: 1950 – 2002.
- Sources: ACI (Automobile Club d'Italia), ISTAT;
- This kind of vehicles suffered a strong reduction during international oil crises supply (1973–1979) that induced a regional economic recession. We introduce an exponential shock through intervention function $x(t) = 1 + c_1 e^{b_1(t-a_1)}$.
- Subsequent recovery of circulating tricars with a maximum located about 1990. Since that year the series exhibits a descent trend, probably due to a technological substitution. Consider a more flexible version of parameter r, namely, a function $r(t) = wt^v$, with $w > 0$ and $v \geq 0$. Obviously, we observe that $r = w$ for $v = 0$.

Italian Tricars Evolution

- Table: Italian tricars stock (1950–2002); BFG + 1 exponential shock and variable exit rate.

	M	w	v	p	q	c_1	b_1	a_1	R^2 (SSE)
BFGe1rwv	817233	1.433E-6	2.596	0.019	0.052	-15.38	-1.59	26.84	0.9813 (15.74E9)

- Table: Italian tricars stock (1950–2002); ARIMA(0,0,4) + BFG + 1 exponential shock and variable exit rate.

Parameter	Estimate	t	P–value
MA(1)	-0.556999	-4.88289	0.000012
MA(2)	-0.542280	-5.95655	0.000000
MA(3)	-0.762372	-8.78945	0.000000
MA(4)	-0.615259	-5.36366	0.000002
PREbe1rwv	0.9770480	24.71490	0.000000
Mean	8242.64	0.588253	0.559121

Estimated white noise variance = 1.73677E8 with 48 degrees of freedom ($R^2 = 0.99011$).

Italian Tricars Evolution

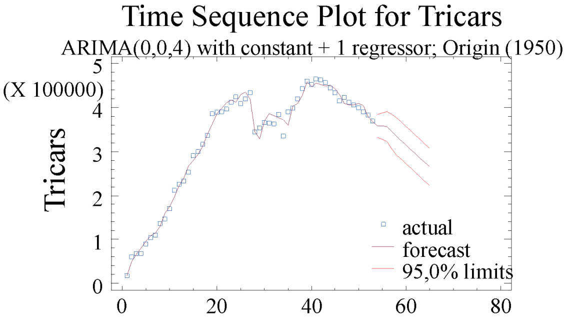

Figure: Italian tricars stock (1950-2002); BFG + 1 shock and variable exit rate; Data sources: ACI and ISTAT.

Summary

- Existence of a clear link between a special class of Cellular Automata and Riccati Equation including Bass family (BM, GBM);
- an answer to the inverse problem (Ganguly et al (2003)): "find a Cellular Automata rule that will have some preselected global properties";
- we avoid Genetic Algorithms which may reach heavy computational complexities without a simple interpretable theoretical framework (see, Venkatesan et al. (2004)).

- Outlook
  - exact solution for variable exit rules $r(t)$;
  - alternative inferential techniques: joint NLS-ARMAX estimation, maximum likelihood, stochastic interventions.