ADVANCED MATHEMATICS FOR STATISTICS- Functional Analysis, A.A. 2017-2018

teacher A. Cesaroni

Exam- November 30, 2017- Time 2 hours

Exercise 1 (9pt)**.** Let
$$f(x) = \frac{1}{1+|x|} \qquad x \in \mathbb{R}^2.$$

(1) Show that $f \in L^\infty(\mathbb{R}^2)$ and compute $\|f\|_\infty$.
(2) Show that $f \notin L^1(\mathbb{R}^2)$.
(3) For which $p > 1$, we have that $f \in L^p(\mathbb{R}^2)$?

Exercise 2 (24pt)**.**

(1) State the Hölder inequality in L^p spaces.

(2) Let $\Omega \subset \mathbb{R}^N$ be a measurable set such that $|\Omega| < \infty$ and $p > 1$. Let (f_n) be a sequence of functions in $L^p(\Omega)$ and $f \in L^p(\Omega)$ such that $f_n \to f$ in $L^p(\Omega)$.
Show that $f_n, f \in L^1(\Omega)$ and that $f_n \to f$ in $L^1(\Omega)$.
Show moreover that $\int_\Omega f_n(x)dx \to \int_\Omega f(x)dx$.
(**hint:** recall that $|\int_\Omega (f_n(x) - f(x))dx| \leq \int_\Omega |f_n(x) - f(x)|dx$.)

(3) State the definition of closed set in a normed space, and recall also the equivalent characterization of closed sets.

(4) Let
$$M = \left\{ f \in L^2(0,1), \;\middle|\; \int_0^1 f(x)dx = 0 \right\}.$$
Show that M is a closed subspace of $L^2(0,1)$ (**hint:** use 2).
Show that for all $f \in L^2(0,1)$, $f - \int_0^1 f(t)dt \in M$.

(5) Show that
$$M^\perp = C = \{f \in L^2(0,1) \; | f(x) = c \text{ almost everywhere for some } c \in \mathbb{R}^n \}$$
(C is the subspace of functions which are constants almost everywhere.)
(**hint:** use 4 and show that if $f \in M^\perp$ then $\int_0^1 (f(x) - \int_0^1 f(t)dt)^2 dx = 0$).

(6) Compute the orthogonal projection of $f(x) = x$ on C.
(7) State the theorem of the orthogonal projection in Hilbert spaces.
(8) Write the decomposition of $f(x) = x$ as the sum of an element of M and of an element of C. Which is the orthogonal projection of f on M?

Sketch of solutions

Solution 1.

(1) We observe that $0 \leq f(x) \leq 1$ for every $x \in \mathbb{R}^2$ and moreover $f(0) = 1$. We consider the set $A(t) = \{t \in \mathbb{R} \mid f(x) > t\}$. Then

$$A(t) = \begin{cases} \emptyset & t \geq 1 \\ B\left(0, \frac{1}{t} - 1\right) & t < 1. \end{cases}$$

In both case, $A(t)$ is a measurable set, so f is measurable. We conclude moreover that $f \in L^\infty(\mathbb{R}^2)$ and $\|f\|_\infty = 1$.

(2) By (1) we get that $|A(t)| = 0$ if $t \geq 1$ and $|A(t)| = \pi\left(\frac{1}{t} - 1\right)^2 = \pi\left(\frac{1}{t^2} - \frac{2}{t} + 1\right)$ if $t < 1$. Therefore

$$\int_{\mathbb{R}^2} f(x)dx = \int_0^1 \pi\left(\frac{1}{t^2} - \frac{2}{t} + 1\right) dt = \left[-\frac{1}{t} - 2\log t + t\right]_0^1 = -\infty.$$

Therefore $f \notin L^1(\mathbb{R}^2)$.

(3) Let $p > 1$ and we compute $A_p(t) = \{t \in \mathbb{R} \mid (f(x))^p > t\}$. Reasoning as in (1) we have that

$$A_p(t) = \begin{cases} \emptyset & t \geq 1 \\ B\left(0, \frac{1}{t^{\frac{1}{p}}} - 1\right) & t < 1. \end{cases}$$

Therefore $|B(t) = 0$ if $t \geq 1$ and $|B(t)| = \pi\left(\frac{1}{t^{\frac{1}{p}}} - 1\right)^2 = \pi\left(\frac{1}{t^{\frac{2}{p}}} - \frac{2}{t^{\frac{1}{p}}} + 1\right)$ if $t < 1$. Then we compute

$$\int_{\mathbb{R}^2} (f(x))^p dx = \int_0^1 \pi\left(\frac{1}{t^{\frac{2}{p}}} - \frac{2}{t^{\frac{1}{p}}} + 1\right) dt = \left[\frac{1}{1-\frac{2}{p}} t^{1-\frac{2}{p}} - \frac{2}{1-\frac{1}{p}} t^{1-\frac{1}{p}} + t\right]_0^1 \neq \infty \qquad \text{iff } p > 2.$$

So $f \in L^p(\mathbb{R}^2)$ for all $p > 2$.

We could also observe that $f(x) \leq \min(\frac{1}{|x|}, 1)$ and that this function is in $L^p(\mathbb{R}^2)$ for all $p > 2$.

Solution 2.

(2) Since $|\Omega| < \infty$, then $\chi_\Omega \in L^q(\mathbb{R}^n)$ for every $q \geq 1$. Then by Hölder inequality if $g \in L^p(\Omega)$ then $g\chi_\Omega \in L^1(\mathbb{R}^n)$, which is equivalent to say that $g \in L^1(\Omega)$. Moreover, always by Hölder inequality

$$\|g\|_{L^1(\Omega)} \leq \|g\|_{L^p(\Omega)} \|\chi_\Omega\|_{L^{\frac{p}{p-1}}(\Omega)} = \|g\|_{L^p(\Omega)} |\Omega|^{\frac{p-1}{p}}.$$

This implies that $f_n, f \in L^1(\Omega)$ for every n and that

$$\|f_n - f\|_{L^1(\Omega)} \leq \|f_n - f\|_{L^p(\Omega)} |\Omega|^{\frac{p-1}{p}}.$$

So, if $\|f_n - f\|_{L^p(\Omega)} \to 0$, then $\|f_n - f\|_{L^1(\Omega)} \to 0$.

Finally observe that

$$|\int_\Omega (f_n(x) - f(x))dx| \leq \int_\Omega |f_n(x) - f(x)|dx.$$

So, if $\int_\Omega |f_n(x) - f(x)|dx \to 0$, then $\int_\Omega (f_n(x) - f(x))dx \to 0$, which was what we wanted to prove.

(4) If $f, g \in M$, then $\alpha f + \beta g \in M$ for every $\alpha, \beta \in \mathbb{R}$, since $\int_0^1 (\alpha f(x) + \beta g(x))dx = \alpha \int_0^1 f(x)dx + \beta \int_0^1 g(x)dx = 0$. This implies that M is a vectorial subspace of $L^2(0,1)$ Moreover, if f_n is a sequence of functions in M such that $f_n \to f$ in $L^2(0,1)$, then by item 2, $0 = \int_0^1 f_n(x)dx \to \int_0^1 f(x)dx$, which implies that $f \in M$. This implies that M is closed.

Observe that if $f \in L^2(0,1)$, then it is immediate to check that $\int_0^1 (f(x) - \int_0^1 f(t)dt)dx = \int_0^1 f(x)dx - \int_0^1 \int_0^1 f(t)dtdx = 0$.

(5) Let $g \in C$, then by definition of C, there exists a constant c such that $g(x) = c$ for all $x \in (0,1) \setminus C_g$, with $|C_g| = 0$. Therefore, for every $f \in M$, we have that

$$\int_0^1 f(x)g(x)dx = \int_{(0,1)\setminus C_g} f(x)cdx = c \int_0^1 f(x)dx = 0.$$

So $C \subseteq M^\perp$.

We consider now $g \in M^\perp$. Then by definition for all $f \in M$, $\int_0^1 f(x)g(x)dx = 0$. Moreover since $f \in M$, we have that

$$\int_0^1 (g(x) - c)f(x)dx = \int_0^1 f(x)g(x)dx - c\int_0^1 f(x)dx = 0 \qquad \text{for all } c \in \mathbb{R}.$$

By 4, we get that $g(x) - \int_0^1 g(t)dt \in M$. So, if we substitute in the previous equality to f the function $g(x) - \int_0^1 g(t)dt \in M$ and to $c = \int_0^1 g(t)dt$, we get that for all $g \in M^\perp$

$$\int_0^1 (g(x) - \int_0^1 g(t)dt)^2 dx = 0.$$

This implies that $g(x) = \int_0^1 g(t)dt$ for almost every x in $(0,1)$, so $g \in C$.

(6) The orthogonal projection of f in C is the element of C which has minimal distance from f (in L^2). So, we compute

$$\min_{c\in\mathbb{R}} \|x - c\|_{L^2(0,1)^2} = \min_{c\in\mathbb{R}} \int_0^1 (x-c)^2 dx = \min_{c\in\mathbb{R}} \int_0^1 x^2 - 2xc + c^2 dx = \min_{c\in\mathbb{R}} (\frac{1}{3} - c + c^2).$$

The minimum is attained at $c = \frac{1}{2}$ so the orthogonal projection of x in C is $\frac{1}{2}$.

(8) By the theorem of orthogonal projection every element $f \in L^2(0,1)$ can be written in a unique way as the such of an element of M and an element of C. By 6, we have that the orthogonal projection of x in C is $\frac{1}{2}$, so $x - \frac{1}{2} \in M$. Therefore the decomposition of x is the following $x = x - \frac{1}{2} + \frac{1}{2}$. The orthogonal projection of x in M is $x - \frac{1}{2}$.