Functional Analysis, A.A. 2018-2019

Exam- December 7- Time 2 hours

Problem 1.

(1) Let $\Omega \subseteq \mathbb{R}^n$ be a open bounded set and $p > 1$. Let $f \in L^p(\Omega)$ and $(f_k)_k$ be a sequence of functions in $L^p(\Omega)$ such that $f_k \to f$ strongly in $L^p(\Omega)$.
Prove that $f_k \to f$ strongly in $L^1(\Omega)$.
Show that
$$\lim_k \int_\Omega f_k(x)dx = \int_\Omega f(x)dx.$$
(2) Write the definition of open set and of closed sets in a metric space and also the equivalent characterization of closed sets. Show that the set
$$E = \left\{ g \in L^p(\Omega) \mid \int_\Omega g(x)dx > 0 \right\}$$
is a open set in $(L^p(\Omega), \| \cdot \|_p)$.
(3) Let $p > 1$, $g \in L^p(\Omega)$ and $(g_k)_k$ be a sequence of functions in $L^p(\Omega)$ such that $g_k \rightharpoonup g$ weakly in $L^p(\Omega)$. Show that for all measurable sets $A \subseteq \Omega$ such that $\mu(A) < +\infty$
$$\lim_k \int_A g_k(x)dx = \int_A g(x)dx.$$

Problem 2.

(1) Let ν be the Gaussian measure, (defined as $\nu(A) = \int_A e^{-|x|^2}dx$). Show that it is absolutely continuous with respect to the Lebesgue measure.
(2) Let δ_0 be the Dirac measure (defined as $\delta_0(A) = 1$ if $0 \in A$ and $\delta_0(A) = 0$ if $0 \notin A$). Show that it is singular with respect to the Lebesgue measure.
(3) Define the sequence of functions, for $k \in \mathbb{N}$,
$$f_k(x) := ke^{-(kx)^2} : \mathbb{R} \to \mathbb{R}.$$
Prove that $f_k \to 0$ in measure in $\mathbb{R}$, and that $f_k \not\to 0$ in $L^p(\mathbb{R})$ for any $p \geq 1$.
(4) Let $a, b \in \mathbb{R}$ with $a < b$. Show that
$$\lim_k \int_\mathbb{R} \chi_{(a,b)}(x) f_k(x)dx = \delta_0((a,b))\nu(\mathbb{R}) = \begin{cases} \nu(\mathbb{R}) & \text{if } a \leq 0 \text{ and } b \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Sketch of solutions

Solution 1.

(1) Since Ω si bounded, by Holder inequality we get that
$$\|f_k - f\|_1 \leq \|f_k - f\|_p \mu(\Omega)^{1-\frac{1}{p}}.$$
Therefore strong L^p convergence implies strong L^1 convergence. Moreover
$$0 \leq \left| \int_\Omega (f_k(x) - f(x))dx \right| \leq \int_\Omega |f_k(x) - f(x)|dx = \|f_k - f\|_1$$
which implies that $\int_\Omega f_k(x)dx \to \int_\Omega f(x)dx$.

(2) A is an open set if for all $x \in A$ there exists $r > 0$ such that $B(x, r) \subseteq A$. C is closed if its complement is open, or equivalently if for all (x_n) sequences such that $x_n \in C$ for all n and $x_n \to x$, we get that $x \in C$. To prove that E is open, it is sufficient to prove that the complement of E which is given by
$$F = \left\{ g \in L^p(\Omega) \mid \int_\Omega g(x)dx \leq 0 \right\}$$
is closed. We take a sequence of functions g_n such that $g_n \in F$ and $g_n \to g$ in $L^p(\Omega)$. Then by the previous item $0 \geq \lim_k \int_\Omega g_k(x)dx = \int_\Omega g(x)dx$. This implies that $g \in F$. So, by the characterization of closed set this implies that F is closed.

(3) If $A \subseteq \Omega$ is a measurable set such that $\mu(A) < +\infty$, then $\chi_A \in L^q(\Omega)$ for all $q \geq 1$. Therefore, by definition of weak convergence $\lim_k \int_\Omega f_k(x)\chi_A(x)dx = \int_\Omega f(x)\chi_A(x)dx$, which is the statement.

Solution 2.

(1) If $A \subseteq \mathbb{R}$ is a measurable set with $\mu(A) = 0$, then $e^{-|x|^2}\chi_A(x) = 0$ almost everywhere. This implies that $\int_\mathbb{R} e^{-|x|^2}\chi_A(x)dx = 0 = \nu(A)$. This implies that $\nu << \mu$.

(2) $\mathbb{R} = (\mathbb{R} \setminus \{0\}) \cup \{0\}$, and $\delta_0(\mathbb{R} \setminus \{0\}) = 0$ whereas $\mu(\{0\}) = 0$.

(3) Fix $\varepsilon > 0$ and let
$$A_\varepsilon^k = \{x \in \mathbb{R} \mid f_k(x) > \varepsilon\} = \{x \mid e^{-k^2|x|^2} > \frac{\varepsilon}{k}\} = \left(-\frac{\sqrt{\log \frac{k}{\varepsilon}}}{k}, \frac{\sqrt{\log \frac{k}{\varepsilon}}}{k} \right).$$
Then $\mu(A_\varepsilon^k) = \frac{2}{k}\sqrt{\log \frac{k}{\varepsilon}} \to 0$ as $k \to +\infty$. This implies that $f_k \to 0$ in measure.

Moreover for every $p \geq 1$, making the change of variable $y = kx\sqrt{p}$,
$$\int_\mathbb{R} k^p e^{-pk^2|x|^2}dx = \frac{k^{p-1}}{\sqrt{p}} \int_\mathbb{R} e^{-|y|^2}dy \not\to 0 \qquad \text{as } k \to +\infty.$$

Therefore $f_k \not\to 0$ in $L^p(\mathbb{R})$.

Finally $\|f_k\|_\infty = k \not\to 0$ as $k \to +\infty$. So $f_k \not\to 0$ in $L^\infty(\mathbb{R})$.

(4) We compute, making the change of variable $y = kx$,

$$\int_{\mathbb{R}} \chi_{(a,b)}(x) f_k(x) dx = \int_a^b k e^{-k^2|x|^2} dx = \int_{ka}^{kb} e^{-|y|^2} dy = \int_{\mathbb{R}} \chi_{(ka,kb)}(y) e^{-|y|^2} dy.$$

Let $y \in \mathbb{R}$ fixed, observe that $\chi_{(ka,kb)}(y) \to 0$ if either $a > 0$ or $b < 0$, whereas $\chi_{(ka,kb)}(y) \to 1$ if $a \leq 0$ and $b \geq 0$. Therefore $\chi_{(ka,kb)}(y)e^{-|y|^2} \to \chi_{(a,b)}(0)e^{-|y|^2}$ and moreover $\chi_{(ka,kb)}(y)e^{-|y|^2} \leq e^{-|y|^2} \in L^1(\mathbb{R})$. Then we conclude by Lebesgue dominated convergence theorem that

$$\lim_k \int_{\mathbb{R}} \chi_{(a,b)}(x) f_k(x) dx = \int_{\mathbb{R}} \chi_{(a,b)}(0) e^{-|y|^2} dy = \chi_{(a,b)}(0)\nu(\mathbb{R}).$$