Main themes

1. Inference under order restrictions

Frequentist approach complex (lack of software)
Bayesian testing natural

2. Marginal models for joint categorical responses

MLE complex. Generalized estimating equations easier
General model for categorical responses

Multivariate generalized linear model

\[ \eta = g(\pi), \quad \eta = X\beta \]

Subject to inequality constraints

\[ U\eta \geq 0, \]

Link function

\[ g(\pi) = C \log(M\pi) \]

Do not assume a parametric model, but test an ordinal structure
Generalized interactions

Non-negative Log odds-ratios Generalized interactions*
local local-local
cumulative local-global
global global-global

* Bartolucci, Colombi, Forcina (2007), St. Sinica.
Marginal model interpretation

Saturated marginal model

\[ \eta = (\lambda_1, \ldots, \lambda_4, \xi, \theta_{11}^G, \ldots, \theta_{14}^G)' = C \log(M \pi) \]

Restriction:
non negative \[ \theta_{11}^G \geq 0, \ldots, \theta_{14}^G \geq 0 \]

global log OR
MLE under inequality constraints

Colombi and Forcina (2001) Biometrika
R package hmmm (Colombi, Giordano, Cazzaro + Lang)

Independence vs Model G
LR Test P-value = 0.021

Independence vs Model L
LR Test P-value = 0.018

Model G vs saturated
LR Test P-value = 0.79

Model L vs saturated
LR Test P-value = 0.46

Using chi-bar-squared. Mixture with weights depending on nuisance parameters. Weights obtained by simulation
Bayesian approach

Specifying prior only for an encompassing model and derive the prior for other models by restrictions

\[ BF_{M_1} = \frac{A_{\text{posterior}}}{A_{\text{prior}}} \]

\[ A_{\text{posterior}} = \text{prop of the parameter space in agreement with } M \text{ under the posterior of the encompassing } m. \]
Bayes factor

Model G
Uniform Dirichlet prior

\[ BF_{G1} = \frac{0.705}{0.2} = 3.5 \]

Model L
Uniform Dirichlet prior

\[ BF_{L1} = \frac{0.014}{0.008} = 1.8 \]

Model L “suspect”
General comments

There is a general framework for marginal models for categorical data, specified by inequality and equality constraints.

Bayesian approach suggested instead of ML if we want to compare models parametrized with different logits

Encompassing priors with point null hypotheses do have problems?
Further comments

Marginal models can be used for joint categorical responses

Agree that there is lack of simple multivariate distribution for categorical responses

Maximizing constrained likelihoods with individual covariates possible (see Evans Forcina 2013 CSDA)
# Starting point

## The trauma data

<table>
<thead>
<tr>
<th></th>
<th>Death</th>
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<th>Major</th>
<th>Minor</th>
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<tr>
<td>High</td>
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## Study of association between Outcome and Treatment
Graphical structure

Regression

Multivariate Regression
MLE under inequality constraints

Colombi and Forcina (2001) Biometrika
R package hmmm (Colombi, Giordano, Cazzaro + Lang)

<table>
<thead>
<tr>
<th>Model G</th>
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<th>Model L</th>
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<tbody>
<tr>
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<td>SE</td>
<td></td>
<td>estimate</td>
<td>SE</td>
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<td>0.0000</td>
<td>0.2620</td>
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</tr>
</tbody>
</table>

Model G vs saturated
LR Test P-value = 0.79

Model L vs saturated
LR Test P-value = 0.46

Using chi-bar-squared. Mixture with weights depending on nuisance parameters. Weights obtained by simulation.
Equality constraints

Point null hypothesis : \( E \eta = 0 \)

Example: \( \theta_{11}^G = \cdots = \theta_{14}^G \)

Model G + equality

<table>
<thead>
<tr>
<th>Obs</th>
<th>Fitted</th>
<th>SE</th>
</tr>
</thead>
<tbody>
<tr>
<td>11</td>
<td>0.2797</td>
<td>0.1434</td>
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</table>

Model L + equality

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<th>SE</th>
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<tr>
<td></td>
<td>-0.0995</td>
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</tr>
</tbody>
</table>

LRT 4.77  4.56 (3 d.f.)
Equality constraints, Bayesian

Point null hypothesis: \( |E \eta| \leq \epsilon \)

e.g., \( |\theta_{11}^G| \leq \epsilon, \ldots, |\theta_{14}^G| \leq \epsilon \)

See Bartolucci, Scaccia Farcomeni (2012)

Problems: rare event simulation overcome by importance-sampling